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Degenerate Sklyanin algebras

Research Article

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Abstract: New trigonometric and rational solutions of the quantum Yang-Baxter equation (QYBE) are obtained by applying some singular gauge transformations to the known Belavin-Drinfeld elliptic R-matrix for $sl(2,\mathbb{C})$. These solutions are shown to be related to the standard ones by the quasi-Hopf twist. We demonstrate that the quantum algebras arising from these new R-matrices can be obtained as special limits of the Sklyanin algebra. A representation for these algebras by the difference operators is found. The $sl(N,\mathbb{C})$-case is discussed.

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1. Introduction

In [1] the equivalence between the elliptic N-particle Calogero-Moser (CM) system and the elliptic Euler-Arnold top was demonstrated using the Hitchin approach to integrable systems. Within this framework the Lax operators arise naturally as the Higgs fields associated with holomorphic bundles of different degrees over an elliptic curve E_τ. The relation between the Lax operators has the form of a gauge transformation:

$$L^{top}_{ell}(z,\tau) = \Xi(z)\, L^{CM}_{ell}(z,\tau)\, \Xi^{-1}(z), \tag{1}$$

where the elements of the matrix $\Xi(z)$ are expressed through theta-functions depending on the coordinates of CM particles. The trigonometric and rational degenerations of (1) are not straightforward and careful analysis is needed. For example, the naive trigonometric limit of (1) leads to divergent expressions because the matrix elements of $\Xi(z)$ are singular in this limit. In [2, 3] we overcame this difficulty by applying a singular gauge transformation depending on τ to (1) before taking the limit:

$$A(\tau)\, L^{top}_{ell}(z,\tau)\, A^{-1}(\tau) = A(\tau)\, \Xi(z)\, L^{CM}_{ell}(z)\, \Xi^{-1}(z)\, A^{-1}(\tau).$$

As a result we obtained the Lax operator for the trigonometric top $L^{top'}_{trig}(z)$ which differs from the standard trigonometric degeneration of the elliptic Lax operator $L^{top}_{trig}(z)$:

$$L^{top'}_{trig}(z) = \lim_{\tau\to i\infty} A(\tau)\, L^{top}_{ell}(z,\tau)\, A^{-1}(\tau),$$
$$L^{top}_{trig}(z) = \lim_{\tau\to i\infty} L^{top}_{ell}(z,\tau).$$

It is well known that the quantization of phase space of the $sl(2,\mathbb{C})$ top is described by the Sklyanin algebra [4].

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It is the algebra with quadratic relations that follow from the Yang-Baxter "RLL" equation with the elliptic Belavin-Drinfeld R-matrix for $sl(2,\mathbb{C})$. The general aim of this paper is to apply the method of singular gauge transformations (developed in [3] for the classical systems) to the quantum case. For example, by means of this approach we obtain the following rational non-dynamical R-matrix:

$$\widetilde{R}^{r}(u)=\begin{bmatrix} \frac{u+2\eta}{2\eta} & 0 & 0 & 0\\ -u\,(u+2\eta)\,\gamma & \frac{u}{2\eta} & 1 & 0\\ -u\,(u+2\eta)\,\gamma & 1 & \frac{u}{2\eta} & 0\\ -u\gamma^{2}\,(u+2\eta)\left(4\eta^{2}+2u\eta+u^{2}\right) & u\,(u+2\eta)\,\gamma & u\,(u+2\eta)\,\gamma & \frac{u+2\eta}{2\eta} \end{bmatrix}. \tag{2}$$

This R-matrix satisfies QYBE for any γ. In the case of $\gamma=0$ we get the ordinary 6-vertex rational R-matrix. The matrix elements of L-operator that satisfies the "RLL" equation with R-matrix (2) obey a quadratic algebra which can be obtained as a degeneration of Sklyanin algebra. It is well known that the quantum algebra arising from the "RLL" equation for the ordinary rational R-matrix (2) at $\gamma=0$ is the Yangian $Y[sl_2]$. Therefore, the rational Sklyanin algebra arising from (2) for generic γ is the one-parametric deformation of $Y[sl_2]$.

The quantum R-matrices play a crucial role in constructing exactly solvable quantum field theories on one-dimensional lattices [14]. For example, the ordinary rational R-matrix (matrix (2) at the point γ=0) defines a spin 1/2 XXX magnetic chain, that is a system of simply interacting spins on the circle. The structure of the R-matrix defines the law of interaction between the spins [13]. The fact that the R-matrix is the solution of QYBE implies that the system possesses internal hidden symmetry that makes this model exactly solvable [14]. The quantum algebras arising from R-matrices are just some algebras with quadratic relations [4], which can be understood as *quantum deformations* of Lie algebras. Their representation theory can be roughly described in physical terms: namely, Hilbert spaces for the spin chains associated with given R-matrix provide the irreducible representations for the quantum algebra. One of the results of the paper is the matrix (2) which generalizes the ordinary XXX spin chain and gives rise to some new integrable model. The hidden symmetry of this system is described by a new quantum algebra (degenerate rational Sklyanin algebra).

The structure of the article is as follows. In Sect. 2 we use singular gauge transformations to obtain new trigonometric and rational solutions of QYBE. In Sect. 3 we apply this technique to L-operators. Sect.4 deals with representations of degenerate Sklyanin algebras by difference operators. In Sect 5 we show that degenerations of the elliptic R-matrix for $sl(2,\mathbb{C})$, that we have found, turn out to be related by a quasi-Hopf twist to the standard degenerations. The $sl(N,\mathbb{C})$-case is discussed in Sect.6.

2. New solutions of QYBE *via* singular gauge transformations

In this section we discuss briefly the standard trigonometric and rational degenerations of the elliptic Baxter's R-matrix for $sl(2,\mathbb{C})$. We give a definition of singular gauge transformations and use them for obtaining new degenerations.

The elliptic R-matrix for $sl(2,\mathbb{C})$ with two dimensional auxiliary space has the form:

$$R^{e}(u)=\sum_{a=0}^{3}W_{a}\,(u+\eta)\;\sigma_{a}\otimes\sigma_{a}, \tag{3}$$

where $W_a(u)=W_a(u|\eta,\tau)$, a=0,...,3 are functions of a variable u (called spectral parameter) with parameters η and τ:

$$W_{a}\,(u)=\frac{\theta_{i(a)}\,(u)}{\theta_{i(a)}\,(\eta)}\,,\qquad i(a)=a+(-1)^{a}.$$

Here, $\theta_a(u)$ are the standard Jacoby theta-functions with characteristics and the modular parameter τ; σ_a are the

Pauli matrices (σ_0 is the unit matrix). For convenience, we write out the explicit form of (3) in terms of the theta-functions with half integer characteristics:

$$R^e(u) = \begin{bmatrix} \frac{\theta_{1,1}(u+\eta,\tau)}{\theta_{1,1}(\eta,\tau)} + \frac{\theta_{1,0}(u+\eta,\tau)}{\theta_{1,0}(\eta,\tau)} & 0 & 0 & \frac{\theta_{0,1}(u+\eta,\tau)}{\theta_{0,1}(\eta,\tau)} - \frac{\theta_{0,0}(u+\eta,\tau)}{\theta_{0,0}(\eta,\tau)} \\ 0 & \frac{\theta_{1,1}(u+\eta,\tau)}{\theta_{1,1}(\eta,\tau)} - \frac{\theta_{1,0}(u+\eta,\tau)}{\theta_{1,0}(\eta,\tau)} & \frac{\theta_{0,1}(u+\eta,\tau)}{\theta_{0,1}(\eta,\tau)} + \frac{\theta_{0,0}(u+\eta,\tau)}{\theta_{0,0}(\eta,\tau)} & 0 \\ 0 & \frac{\theta_{0,1}(u+\eta,\tau)}{\theta_{0,1}(\eta,\tau)} + \frac{\theta_{0,0}(u+\eta,\tau)}{\theta_{0,0}(\eta,\tau)} & \frac{\theta_{1,1}(u+\eta,\tau)}{\theta_{1,1}(\eta,\tau)} - \frac{\theta_{1,0}(u+\eta,\tau)}{\theta_{1,0}(\eta,\tau)} & 0 \\ \frac{\theta_{0,1}(u+\eta,\tau)}{\theta_{0,1}(\eta,\tau)} - \frac{\theta_{0,0}(u+\eta,\tau)}{\theta_{0,0}(\eta,\tau)} & 0 & 0 & \frac{\theta_{1,1}(u+\eta,\tau)}{\theta_{1,1}(\eta,\tau)} + \frac{\theta_{1,0}(u+\eta,\tau)}{\theta_{1,0}(\eta,\tau)} \end{bmatrix}.$$

The elliptic R-matrix (3) satisfies QYBE:

$$R^e_{12}(u-v)\, R^e_{13}(u)\, R^e_{23}(v) = R^e_{23}(v)\, R^e_{13}(u)\, R^e_{12}(u-v). \tag{4}$$

Here R-matrix $R_{12}(u)$ acts in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ as $R(u)$ on the first and second spaces, and as the identity operator on the third one (similarly for R_{13} and $R_{2,3}$). Solutions of (4) play an important role for the construction of solvable two-dimensional statistical models [5], in the integrable systems [6], and representation theory [7]. Letting the modular parameter to imaginary infinity $\tau \to \mathrm{i}\infty$ we get the standard trigonometric R-matrix:

$$R^t(u) = \frac{1}{\sin(2\pi\eta)} \begin{bmatrix} \sin(\pi\,(u+2\eta)) & 0 & 0 & 0 \\ 0 & \sin(\pi u) & \sin(2\pi\eta) & 0 \\ 0 & \sin(2\pi\eta) & \sin(\pi u) & 0 \\ 0 & 0 & 0 & \sin(\pi\,(u+2\eta)) \end{bmatrix}. \tag{5}$$

To get the standard rational degeneration of (5), let us substitute in (5) the constant π by a formal variable x. In the limit $x = 0$ we get:

$$R^r(u) = \frac{1}{2\eta} \begin{bmatrix} u+2\eta & 0 & 0 & 0 \\ 0 & u & 2\eta & 0 \\ 0 & 2\eta & u & 0 \\ 0 & 0 & 0 & u+2\eta \end{bmatrix}. \tag{6}$$

On the set of solutions of QYBE for $sl(2,\mathbb{C})$ acts the group $SL(2,\mathbb{C})$ (the gauge group). More precisely, let $G \in SL(2,\mathbb{C})$ be two-dimensional matrix. We say that R-matrix R_2 is the gauge transformation of R_1 by matrix G if:

$$R_2 = A\,R_1\,A^{-1}, \qquad A = G_1\,G_2, \qquad G_1 = G \otimes 1, \qquad G_2 = 1 \otimes G.$$

The matrices related by a gauge transformation are called gauge equivalent.

Let us consider the gauge transformation depending on modular parameter $q = \exp(2\pi i \tau)$ with the matrix:

$$G^t = \begin{bmatrix} q^{1/8}\,\alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2}\,q^{-1/8} \end{bmatrix}. \tag{7}$$

If we take the trigonometric limit $q = 0$ after applying gauge transformation (7) to elliptic R-matrix (3) we get the following result:

$$\widetilde{R}^t(u) = \frac{1}{\sin(2\pi\eta)} \begin{bmatrix} \sin(\pi\,(u+2\eta)) & 0 & 0 & 0 \\ 0 & \sin(\pi u) & \sin(2\pi\eta) & 0 \\ 0 & \sin(2\pi\eta) & \sin(\pi u) & 0 \\ 4\,\alpha^2 \sin(\pi u)\sin(2\pi\eta)\sin(\pi\,(u+2\eta)) & 0 & 0 & \sin(\pi\,(u+2\eta)) \end{bmatrix}. \tag{8}$$

This matrix, obviously, satisfies QYBE because elliptic R-matrix (3), gauge transformed by (7), satisfies QYBE for any q. It can be shown that R-matrix (8) is not gauge equivalent to (5), therefore it defines a new trigonometric solution of QYBE. Moreover, this R-matrix is the one-parametric generalization of standard one (5) with the parameter α. At the point $\alpha = 0$ we get the standard trigonometric degeneration.
We note that the determinant of the matrix G^t is equal to one for any value of the parameter q, *i.e.* for any q this matrix belongs to the gauge group $SL(2, \mathbb{C})$. This matrix, however, is singular at the point $q = 0$ and this is why we obtain a new trigonometric R-matrix that is not gauge equivalent to the standard trigonometric degeneration (5). We call gauge transformations with these properties *singular*. The importance of these transformations lies in the fact that, as in the examples above, they can be used for obtaining new degenerate solutions of QYBE.
Let us consider another example of a singular gauge transformation with the matrix:

$$G^r = \begin{bmatrix} x\,\alpha^{1/2}\,\beta^{-1/2} & 0 \\ 2\,\alpha^{1/2}\,\beta^{1/2}\,x^{-1} & \beta^{1/2}\,\alpha^{-1/2}\,x^{-1} \end{bmatrix}. \tag{9}$$

The determinant of this matrix is equal to one for any value of the parameter x (*i.e.* G^r belongs to the gauge group), but, at the point $x = 0$ G^r, becomes singular. To get a rational degeneration of (8) we should substitute the constant π by a formal variable x. Then, applying the gauge transformation with matrix (9) to (8) and taking the limit $x = 0$ we get the following rational R-matrix:

$$\widetilde{R}^r(u) = \begin{bmatrix} \frac{u+2\eta}{2\eta} & 0 & 0 & 0 \\ -u\,(u+2\eta)\,\beta & \frac{u}{2\eta} & 1 & 0 \\ -u\,(u+2\eta)\,\beta & 1 & \frac{u}{2\eta} & 0 \\ -u\beta^2\,(u+2\eta)\left(4\,\eta^2+2\,u\eta+u^2\right) & u\,(u+2\eta)\,\beta & u\,(u+2\eta)\,\beta & \frac{u+2\eta}{2\eta} \end{bmatrix}. \tag{10}$$

This R-matrix is the one-parametric generalization of standard rational R-matrix (6): it satisfies QYBE for any value of the parameter β, and at the point $\beta = 0$ coincides with (6). It should also be mentioned, that the form of the R-matrix (10) is similar to one developed in [8–10]. In this way, some relation between our approach and Jordanian transformation appeared in [8–10] should exist.
We should mention that matrices of gauge transformations (7) and (9) have special dependence on their parameters q and x: after applying these transformations we get finite limits at $q = 0$ and $x = 0$. This imposes strict conditions on the matrix elements of (7) and (9).

3. Quantum Lax operators

The universal elliptic L-operator with 2-dimensional auxiliary space, has the form:

$$L^e\,(u) = \sum_{a=0}^{3} W_a\,(u)\;S_a \otimes \sigma_a. \tag{11}$$

Let us write out the explicit expression of this operator in terms of the elliptic theta-functions with half-integer characteristics:

$$L^e(u) = \begin{bmatrix} \frac{\theta_{1,1}(u)}{2\,\theta_{1,1}(\eta)}\,S_0 - \frac{\theta_{1,0}(u)}{2\,\theta_{1,0}(\eta)}\,S_3 & \frac{\theta_{0,1}(u)}{2\,\theta_{0,1}(\eta)}\,S_1 + \frac{i\theta_{0,0}(u)}{2\,\theta_{0,0}(\eta)}\,S_2 \\ \frac{\theta_{0,1}(u)}{\theta_{0,1}(2\,\eta)}\,S_1 - \frac{i\theta_{0,0}(u)}{2\,\theta_{0,0}(\eta)}\,S_2 & \frac{\theta_{1,1}(u)}{2\,\theta_{1,1}(\eta)}\,S_0 + \frac{\theta_{1,0}(u)}{2\,\theta_{1,0}(\eta)}\,S_3 \end{bmatrix}. \tag{12}$$

Here the operators S_0, S_α $\alpha = 1, 2, 3$, are the generators of Sklyanin algebra [4]. These operators obey the conditions under which L-operator (12) satisfies the so called "RLL" equation:

$$R^e_{1,2}(u-v)\, L^e_1(u)\, L^e_2(v) \;=\; L^e_2(v)\, L^e_1(u)\, R^e_{1,2}(u-v), \tag{13}$$

where we use the following standard notations: $L_1(u) = L(u) \otimes 1$ and $L_2(u) = 1 \otimes L(u)$.
Our aim is to find the form of the trigonometric and rational L-operators that satisfy the "RLL" equation for R-matrices (8) and (10). To find these operators we should apply gauge transformations (7) and (9) to elliptic L-operator (12). A gauge transformation with matrix G acts on Lax operators as a conjugation in the "external" space:

$$L^{'}(u) = G\, L(u)\, G^{-1} \tag{14}$$

and as a change of the generators S_i by new ones $S_i^{'}$ in the "internal" space:

$$\begin{bmatrix} S_0^{'} - S_3^{'} & S_1^{'} + iS_2^{'} \\ S_1^{'} - iS_2^{'} & S_0^{'} + S_3^{'} \end{bmatrix} = G \begin{bmatrix} S_0 - S_3 & S_1 + iS_2 \\ S_1 - iS_2 & S_0 + S_3 \end{bmatrix} G^{-1}. \tag{15}$$

Let us consider how this scheme works in the trigonometric and rational cases. The change of the generators induced by gauge transformation (15) with matrix (7) has the form:

$$\begin{bmatrix} S_0^t - S_3^t & S_1^t + iS_2^t \\ S_1^t - iS_2^t & S_0^t + S_3^t \end{bmatrix} = \begin{bmatrix} q^{1/8}\,\alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2}\, q^{-1/8} \end{bmatrix} \begin{bmatrix} S_0^e - S_3^e & S_1^e + iS_2^e \\ S_1^e - iS_2^e & S_0^e + S_3^e \end{bmatrix} \begin{bmatrix} q^{1/8}\,\alpha^{-1/2} & 0 \\ 0 & \alpha^{1/2}\, q^{-1/8} \end{bmatrix}^{-1}$$

or more explicitly:

$$\begin{aligned} S_0^e &= S_0^t, \\ S_3^e &= S_3^t, \\ S_1^e &= \frac{\alpha}{2i}\, q^{-1/4} \left(S_1^t + i\, S_2^t \right) + \frac{1}{2\, i\, \alpha}\, q^{1/4} \left(S_1^t - i\, S_2^t \right), \\ S_2^e &= \frac{\alpha}{2}\, q^{-1/4} \left(S_1^t + i\, S_2^t \right) + \frac{1}{2\, \alpha}\, q^{1/4} \left(S_1^t - i\, S_2^t \right). \end{aligned} \tag{16}$$

Substituting these expressions into $G^t\, L^e(u)\, (G^t)^{-1}$ and taking the limit $q \to 0$ we get the following trigonometric Lax operator:

$$L^t(u) = \begin{bmatrix} S_0^t \sin(\pi\, u) \cot(\pi\, \eta) - S_3^t \sin(\pi\, u) & S_1^t + iS_2^t \\ \alpha^2 \left(S_1^t + iS_2^t\right) \left(\cos(2\,\pi\, u) - \cos(2\,\pi\,\eta)\right) + S_1^t - iS_2^t & S_0^t \sin(\pi\, u) \cot(\pi\,\eta) + S_3^t \sin(\pi\, u) \end{bmatrix}. \tag{17}$$

This Lax operator is a one-parametric deformation of the standard trigonometric degeneration of (12). It coincides with the standard trigonometric L-operator at the point $\alpha = 0$.
Let us move on to the rational case. The change of the generators of the algebra induced by gauge transformation (9) has the form:

$$\begin{bmatrix} S_0^r - S_3^r & S_1^r + iS_2^r \\ S_1^r - iS_2^r & S_0^r + S_3^r \end{bmatrix} = \begin{bmatrix} x\,\alpha^{1/2}\,\gamma^{-1/2} & 0 \\ 2\,\alpha^{1/2}\,\beta^{1/2}\,x^{-1} & \beta^{1/2}\,\alpha^{-1/2}\,x^{-1} \end{bmatrix} \begin{bmatrix} S_0^t - S_3^t & S_1^t + iS_2^t \\ S_1^t - iS_2^t & S_0^t + S_3^t \end{bmatrix} \begin{bmatrix} x\,\alpha^{1/2}\,\beta^{-1/2} & 0 \\ 2\,\alpha^{1/2}\,\beta^{1/2}\,x^{-1} & \beta^{1/2}\,\alpha^{-1/2}\,x^{-1} \end{bmatrix}^{-1}.$$

Or more explicitly:

$$
\begin{aligned}
S_0^t &= S_0^r, \\
S_1^t &= -\frac{\left(-\beta^2 - x^4\alpha^2 + 4\,\beta^2\alpha^2\right) S_1^r}{2\,\beta\, x^2\alpha} - \frac{\left(-i\beta^2 + ix^4\alpha^2 + 4\,i\beta^2\alpha^2\right) S_2^r}{2\,\beta\, x^2\alpha} + 2\,\alpha\, S_3^r, \\
S_2^t &= \frac{-1/2\, i\left(\beta^2 - x^4\alpha^2 + 4\,\beta^2\alpha^2\right) S_1^r}{\beta\, x^2\alpha} - \frac{1/2\, i\left(i\beta^2 + ix^4\alpha^2 + 4\,i\beta^2\alpha^2\right) S_2^r}{\beta\, x^2\alpha} + 2\, i\,\alpha\, S_3^r, \\
S_3^t &= -2\,\frac{S_1^r\beta}{x^2} - \frac{2\, i\, S_2^r\beta}{x^2} + S_3^r.
\end{aligned}
\tag{18}
$$

To get the rational limit, in complete analogy with the previous section, we should substitute in (17) the constant π by a formal variable x. Then, substituting (18) into $G^r\, L^t(u)\,(G^r)^{-1}$ and taking the limit $x = 0$ we get the following rational L-operator:

$$
L^r(u) = \begin{bmatrix} L^r_{1,1} & L^r_{1,2} \\ L^r_{2,1} & L^r_{2,2} \end{bmatrix},
$$

where:

$$
L^r_{1,1} = \frac{1}{2\,\eta}\left(-S_1^r\eta\, u^2 - i\, S_2^r u^2\eta + i\, S_2^r\eta^3 + S_1^r\eta^3\right)\beta - \frac{1}{2\,\eta}\left(S_3^r\eta - S_0^r u\right),
$$

$$
L^r_{1,2} = 1/2\, S_1^r + 1/2\, i\, S_2^r,
$$

$$
L^r_{2,1} = \left(3/2\, i\eta^4 S_2^r - i\eta^2 u^2 S_2^r - 1/2\, iu^4 S_2^r - 1/2\, u^4 S_1^r - u^2 S_1^r\eta^2 + 3/2\, S_1^r\eta^4\right)\beta^2 + \left(S_2^r u^2 - S_2^r\eta^2\right)\beta - 1/2\, iS_2^r + 1/2\, S_1^r,
$$

$$
L^r_{2,2} = 1/2\,\frac{\left(-S_1^r\eta^3 + i\, S_2^r u^2\eta - i\, S_2^r\eta^3 + S_1^r u^2\eta\right)\beta}{\eta} + 1/2\,\frac{S_3^r\eta + S_0^r u}{\eta}.
$$

Let us introduce the trigonometric and rational Sklyanin algebras as algebras with four generators S_i^t and S_i^r obeying relations following from the "RLL" equations:

$$
R^t_{1,2}\,(u - v)\; L_1^t(u)\, L_2^t(v) \;=\; L_2^t(v)\, L_1^t(u)\, R^t_{1,2}(u - v), \tag{19}
$$

$$
R^r_{1,2}\,(u - v)\; L_1^r(u)\, L_2^r(v) \;=\; L_2^r(v)\, L_1^r(u)\, R^r_{1,2}(u - v). \tag{20}
$$

In sections 4 and 5 we study these algebras. We write out the relations for S_i^t and S_i^r explicitly and find their representations in terms of the difference operators acting on the space of meromorphic functions on one variable.

4. Degenerate Sklyanin algebras

Sklyanin algebra [4] is the algebra with four generators S_i^e, $i = 0, ..., 3$ and the following quadratic relations:

$$
[S_i^e, S_j^e]_- = i\,[S_0^e, S_k^e]_+, \qquad [S_0^e, S_k^e]_- = i\, J_{i,j}\,[S_i^e, S_j^e]_+, \qquad J_{i,j} = \frac{J_j - J_i}{J_k}; \tag{21}
$$

where $[A, B]_\pm = A\,B \pm B\,A$. A triple of indices (i, j, k) in (21) stands for any cyclic permutation of $(1, 2, 3)$. Structure constants J_i have the form:

$$
J_1 = \frac{\theta_{0,1}\,(2\,\eta)\,\theta_{0,1}\,(0)}{\theta_{0,1}\,(\eta)^2}, \qquad J_2 = \frac{\theta_{0,0}\,(2\,\eta)\,\theta_{0,0}\,(0)}{\theta_{0,0}\,(\eta)^2}, \qquad J_3 = \frac{\theta_{1,0}\,(2\,\eta)\,\theta_{1,0}\,(0)}{\theta_{1,0}\,(\eta)^2}. \tag{22}
$$

Relations (21) were introduced by E.Slkyanin as the minimal set of conditions under which L-operator (11) satisfies the "RLL" equation with elliptic R-matrix (3). This algebra possesses two independent central elements (the Casimir elements):

$$\Omega_1^e = S_0^{e\,2} + S_1^{e\,2} + S_2^{e\,2} + S_3^{e\,2}\,, \qquad \Omega_2^e = J_1\, S_1^{e\,2} + J_2\, S_2^{e\,2} + J_3\, S_3^{e\,2}. \tag{23}$$

In [4] the following representation of Sklyanin algebra in terms of the difference operators was found:

$$\begin{gathered} S_i^e = \frac{\chi_i(u - s\,\eta)}{\theta_{1,1}\,(2\,u)}\, \exp(\eta\,\partial_u) - \frac{\chi_i(-u - s\,\eta)}{\theta_{1,1}\,(2\,u)}\, \exp(-\eta\,\partial_u), \\ \chi_0 = \theta_{1,1}\,(2\,u)\;\theta_{1,1}\,(\eta)\,, \qquad \chi_1 = \theta_{0,1}\,(2\,u)\;\theta_{0,1}\,(\eta)\,, \qquad \chi_2 = \theta_{0,0}\,(2\,u)\;\theta_{0,0}\,(\eta)\,, \qquad \chi_3 = \theta_{1,0}\,(2\,u)\;\theta_{1,0}\,(\eta) \end{gathered} \tag{24}$$

with standard notations $\exp(\pm\eta\,\partial_u)(f)\,(u) = f\,(u \pm \eta)$. In this representation, central elements (23) are given by scalar operators:

$$\Omega_1^e = 4\,\theta_{1,1}^2\,(\,(2\,s + 1)\,\eta\,)\,, \qquad \Omega_2^e = 4\,\theta_{1,1}\,(\,2\,(s + 1)\,\eta\,)\;\theta_{1,1}\,(\,2\,\eta\,)\,. \tag{25}$$

Let us consider the degenerations of the Sklyanin algebra related to trigonometric and rational R-matrices (8) and (10). Gauge transformation (7), as shown above, induces the transformation for the generators of Sklyanin algebra (16). Substituting these expressions into quadratic relations (21) and expanding them in powers of q up to the first non-trivial order, we have:

$$\begin{aligned} &[\,S_i^t,\,S_j^t\,]_- = i\,[\,S_0^t,\,S_k^t\,]_+, \\ &[\,S_0^t,\,S_1^t\,]_- = \frac{C_2}{4}\,[\,S_1^t,\,S_3^t\,]_+ + \frac{i\,(2\,C_1 - C_2)}{4}\,[\,S_2^t,\,S_3^t\,]_+, \\ &[\,S_0^t,\,S_2^t\,]_- = -\frac{i\,(C_2 + 2\,C_1)}{4}\,[\,S_1^t,\,S_3^t\,]_+ - \frac{C_2}{4}\,[\,S_2^t,\,S_3^t]_+, \\ &[\,S_0^t,\,S_3^t\,]_- = \frac{C_3}{2}\,\Big(\,S_1^{t\,2} - S_2^{t\,2} - i\,[\,S_1^t,\,S_2^t\,]_+\,\Big)\,, \end{aligned} \tag{26}$$

where the constants C_1, C_2, and C_3 have the form:

$$C_1 = \lim_{q\to 0}\big(J_{3,1} - J_{1,2}\big) = -2\,\frac{\sin\,(\pi\,\eta)^2}{\cos\,(\pi\,\eta)^2}, \tag{27}$$

$$C_2 = \lim_{q\to 0}\big(J_{1,2} + J_{3,1}\big) = -16\,\frac{\sin\,(\pi\,\eta)^2\cos\,(2\,\pi\,\eta)}{\cos\,(\pi\,\eta)^2}, \tag{28}$$

$$C_3 = \lim_{q\to 0}\frac{J_{2,3}}{q} = 4\,\sin\,(2\,\pi\,\eta)^2\,. \tag{29}$$

The algebra generated by four elements S_i^t, $i = 0, ..., 3$ with relations (26) represent a natural trigonometric degeneration of the Sklyanin algebra, and we call it *trigonometric Sklyanin algebra*. This algebra possesses two independent Casimirs:

$$\Omega_1^t = S_0^{t\,2} + S_1^{t\,2} + S_2^{t\,2} + S_3^{t\,2}\,, \qquad \Omega_2^t = -S_1^{t\,2} - S_2^{t\,2} - L_1\,S_3^{t\,2} - \frac{L_3}{4}\,\Big(S_1^{t\,2} - S_2^{t\,2} - i\,[S_1^t, S_2^t]_+\Big) \tag{30}$$

with the following constants:

$$L_1 = \frac{\cos\,(2\,\pi\,\eta)}{\cos\,(\pi\,\eta)^2}, \qquad L_3 = 48\,\cos\,(\pi\,\eta)^2 - 16 - 32\,\cos\,(\pi\,\eta)^4\,.$$

Substituting the generators (24) into (16) and taking the limit $q = 0$ we find the following representations for the generators of the trigonometric Sklyanin algebra:

$$S_0^t = \frac{-\cos(\pi\ (2u-\eta-2s\eta)) + \cos(\pi\ (2u+\eta-2s\eta))}{\sin(2\pi u)} \exp(\eta\,\partial_u) + \frac{\cos(\pi\ (\eta+2u+2s\eta)) - \cos(\pi\ (-\eta+2u+2s\eta))}{\sin(2\pi u)} \exp(-\eta\,\partial_u),$$

$$S_1^t = \frac{1-2\cos(2\pi\eta) - 2\cos(4\pi\ (u+s\eta))}{2\sin(2\pi u)} \exp(-\eta\,\partial_u) - \frac{1-2\cos(2\pi\eta) - 2\cos(4\pi\ (-u+s\eta))}{2\sin(2\pi u)} \exp(\eta\,\partial_u),$$

$$S_2^t = \frac{i\,(2\cos(2\pi\eta) + 2\cos(4\pi\ (u+s\eta)) + 1)}{2\sin(2\pi u)} \exp(-\eta\,\partial_u) - \frac{i\,(2\cos(2\pi\eta) + 2\cos(4\pi\ (-u+s\eta)) + 1)}{2\sin(2\pi u)} \exp(\eta\,\partial_u),$$

$$S_3^t = -\frac{\cos(\pi\ (-\eta+2u+2s\eta)) + \cos(\pi\ (\eta+2u+2s\eta))}{\sin(2\pi u)} \exp(-\eta\,\partial_u) + \frac{\cos(\pi\ (-\eta-2u+2s\eta)) + \cos(\pi\ (\eta-2u+2s\eta))}{\sin(2\pi u)} \exp(\eta\,\partial_u).$$

Substituting these difference operators into (30) we find that in this representation the Casimir operators have the values:

$$\Omega_1^t = 16\ (\sin(\eta\pi\ (1+2s)))^2, \qquad \Omega_2^t = 16\sin(2\pi\eta\ (s+1))\sin(2\pi\eta s)\,. \tag{31}$$

We should note here, that this trigonometric degeneration of the Sklyanin algebra appeared for the first time in [11]. Our approach differs from [11] by other choice of the generators for the degenerate algebra.
To find a rational degeneration, we note that gauge transformation (9) induces the transformation for the generators of trigonometric Sklyanin algebra (18). Substituting (18) to the relations (26) and taking the limit $x = 0$ (we replace everywhere the constant π by a formal parameter x) we get the following relations for the generators of *the rational Sklyanin algebra*:

$$\begin{gathered}
[S_1^r, S_2^r]_- = i\,[S_0^r, S_3^r]_+, \qquad [S_2^r, S_3^r]_- = i\,[S_0^r, S_1^r]_+, \qquad [S_3^r, S_1^r]_- = i\,[S_0^r, S_2^r]_+, \\
[\,S_0^r,\ S_3^r,\,]_- = 16\,i\,\eta^4\,[S_1^r, S_3^r]_+ - 2\,i\,\eta^2[S_2^r, S_3^r]_+ - 16\,\eta^4\ \left(S_1^{r\,2} - S_2^{r\,2}\right), \\
[S_0^r, S_2^r]_- = -8\,\eta^4\,[S_2^r, S_3^r]_+ + 2\ \left(8\,\eta^4 - 1\right)\ \eta^2\,[S_1^r, S_2^r]_+ - 8\,i\,\eta^4\,[S_1^r, S_3^r]_+ \\
+4\,i\,\eta^2\ \left(4\,\eta^4 - 1\right)\ S_1^{r\,2} - 16\,i\,\eta^6\,S_2^{r\,2} + 4\,i\,\eta^2\,S_3^{r\,2}, \\
[S_0^r, S_1^r]_- = -8\,i\,\eta^4\,[S_2^r, S_3^r]_+ + 2\,i\ \left(8\,\eta^4 + 1\right)\ \eta^2\,[S_1^r, S_2^r]_+ + 8\,\eta^2\,[S_1^r, S_3^r]_+ \\
-16\,\eta^6\,S_1^{r\,2} + 4\ \left(4\,\eta^4 + 1\right)\ \eta^2\,S_2^{r\,2} - 4\,\eta^2\,S_3^{r\,2}.
\end{gathered} \tag{32}$$

The central elements of this algebra have the form:

$$\begin{gathered}
\Omega_1^r = S_0^{r\,2} + S_1^{r\,2} + S_2^{r\,2} + S_3^{r\,2}, \\
\Omega_2^r = \left(1 - 12\,\eta^4\right)\ S_1^{r\,2} + \left(12\,\eta^4 + 1\right)\ S_2^{r\,2} + S_3^{r\,2} + 12\,i\,\eta^4\,[S_1^r, S_2^r]_+ + 2\,\eta^2\,[S_1^r, S_3^r]_+ - 2\,i\,\eta^2\,[S_2^r, S_3^r]_+.
\end{gathered} \tag{33}$$

As well as in the trigonometric case, we find the difference operators which represent the generators of this algebra:

$$S_0^r = \frac{\left(-2\,u\eta - 2\,s\eta^2\right)}{u} \exp(-\eta\,\partial_u) + \frac{\left(-2\,u\eta + 2\,s\eta^2\right)}{u} \exp(\eta\,\partial_u),$$

$$S_1^r = \frac{\left(8\,s^2\eta^4 - 64\,u^3 s\eta + 16\,us\eta^3 - 64\,us^3\eta^3 - 96\,u^2 s^2\eta^2 - \eta^4 + 8\,u^2\eta^2 - 16\,u^4 - 16\,s^4\eta^4 + 1\right)}{4\,u}\exp(-\eta\,\partial_u) + \frac{\left(16\,s^4\eta^4 + \eta^4 - 64\,u^3 s\eta + 16\,u^4 - 64\,us^3\eta^3 + 96\,u^2 s^2\eta^2 - 1 - 8\,s^2\eta^4 - 8\,u^2\eta^2 + 16\,us\eta^3\right)}{4\,u}\exp(\eta\,\partial_u),$$

$$S_2^r = \frac{\left(-8\,is^2\eta^4 + i + 64\,is^3\eta^3 u - 16\,is\eta^3 u + 16\,is^4\eta^4 + i\eta^4 + 96\,iu^2 s^2\eta^2 - 8\,iu^2\eta^2 + 64\,iu^3 s\eta + 16\,iu^4\right)}{4\,u}\exp(-\eta\,\partial_u) + \frac{\left(64\,is^3\eta^3 u + 8\,iu^2\eta^2 - 96\,iu^2 s^2\eta^2 - 16\,is\eta^3 u + 8\,is^2\eta^4 - i\eta^4 + 64\,iu^3 s\eta - 16\,is^4\eta^4 - i - 16\,iu^4\right)}{4\,u}\exp(\eta\,\partial_u),$$

$$S_3^r = \frac{\left(\eta^2 + 4\,u^2 + 8\,us\eta + 4\,s^2\eta^2\right)}{2\,u}\exp(-\eta\,\partial_u) + \frac{\left(-4\,u^2 + 8\,us\eta - \eta^2 - 4\,s^2\eta^2\right)}{u}\exp(\eta\,\partial_u).$$

Substituting these generators into the expressions for the central elements, we find that in this representation they acts as scalar operators with the following eigenvalues:

$$\Omega_1^r = 16\,\eta^2\,(2\,s+1)^2\,, \qquad \Omega_2^r = 64\,\eta^2 s\,(1+s)\,. \tag{34}$$

5. Quasi-Hopf twist

In this section we demonstrate that standard trigonometric and rational R-matrices (5) and (6) are related by the quasi-Hopf twist to non-standard degenerations (8) and (10). To show this, we construct the matrices of quasi-Hopf twists for these cases explicitly.

We start from the trigonometric case. Let us consider the operator acting in $\mathbb{C}^2\otimes\mathbb{C}^2$, with the matrix:

$$Q^t(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2\,\alpha^2\sin(\pi\,u)\sin(2\,\pi\,\eta) & 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

This operator has the following properties:

$$Q^t(u)^{-1} = Q^t(-u), \qquad P\,Q^t(u)\,P = Q^t(u), \tag{36}$$

where P is the permutation operator in $\mathbb{C}^2\otimes\mathbb{C}^2$. We find that trigonometric R-matrices (8) and (5) are related by this operator in the following way:

$$\widetilde{R}^t(u) = Q^t(u)\,R^t(u)\,Q^t(u). \tag{37}$$

Using (36) we can rewrite this relation in the form of a quasi-Hopf twist:

$$\widetilde{R}^t(u-v)\,F^t_{21}(v,u) = F^t_{12}(u,v)\,R^t(u-v), \tag{38}$$

where

$$F^t_{12}(u,v) = Q^t(u-v), \qquad F^t_{21}(u,v) = P\,F^t_{12}(u,v)\,P,$$

Similarly, in the rational case, we find the following matrix:

$$Q^r(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -u\,\beta\,\eta & 1 & 0 & 0 \\ -u\,\beta\,\eta & 0 & 1 & 0 \\ -\eta u^3\,\beta^2 - \eta^2\,u^2\,\beta^2 - 4\,\eta^3\,u\,\beta^2 & u\,\beta\,h & u\,\beta\,h & 1 \end{bmatrix}. \tag{39}$$

This operator has the following properties:

$$Q^r(u)^{-1} = Q^r(-u), \qquad P\,Q^r(u)\,P = Q^r(u). \tag{40}$$

Using this matrix, we can relate rational R-matrices (10) and (6) in the following way:

$$\widetilde{R}^r(u) = Q^r(u)\,R^r(u)\,Q^r(u). \tag{41}$$

Similar to the trigonometric case, using (40) we can rewrite this equation in the form of a quasi-Hopf twist:

$$\widetilde{R}^r(u-v)\,F^r_{21}(v,u) = F^r_{12}(u,v)\,R^r(u-v), \tag{42}$$

where

$$F^r_{12}(u,v) = Q^r(u-v), \quad F^r_{21}(u,v) = P\,F^r_{12}(u,v)\,P.$$

Therefore, equations (38) and (42) show that the standard and nonstandard degenerate R-matrices turn out to be related by the quasi-Hopf twists F^t_{12} and F^r_{12}.

6. $sl(N,\mathbb{C})$-case

In this section we obtain non-standard trigonometric and rational degenerations of the elliptic Belavin's R-matrix for $sl(N,\mathbb{C})$. The elliptic R-matrix for $sl(N,\mathbb{C})$ has the following form:

$$R^e(u) = \sum_{i',j'=1}^{N} \delta_{|i+j||i'+j'|} \frac{\theta^{(i'-j')}(u+2\eta)\,\theta^{(0)}(u)}{\theta^{(i'-i)}(2\eta)\,\theta^{(i-j')}(u)} E_{i\,i'} \otimes E_{j\,j'}, \tag{43}$$

where $(E_{i\,i'})_{k\,l} = \delta_{i\,k}\,\delta_{i'\,l}$ are elements of standard basis in the space of $N^2 \times N^2$ matrices, $|i| = i \mod N$, and we use the following notations for the theta-functions:

$$\theta^{(j)}(u) = \sum_{m\in\mathbb{Z}} \exp\left(\pi\, i\, N\,\tau\left(m+\frac{1}{2}-\frac{j}{N}\right)^2 + 2\,\pi\, i\left(m+\frac{1}{2}-\frac{j}{N}\right)\left(u+\frac{1}{2}\right)\right). \tag{44}$$

Similarly to the $N=2$ case, to get the non-standard degenerations, we should apply the additional gauge transformations before taking the corresponding limits. Given the matrices of these transformations $G^t(q)$ and $G^r(x)$, we can find the nonstandard degenerations by calculating the following limits:

$$\widetilde{R}^t(u) = \lim_{q\to 0} G^t_1(q)\,G^t_2(q)\,R^e(u)\,(G^t_1(q)\,G^t_2(q))^{-1}, \tag{45}$$

$$\widetilde{R}^r(u) = \lim_{x\to 0} G^r_1(x)\,G^r_2(x)\,R^t(u)\,(G^r_1(x)\,G^r_2(x))^{-1}. \tag{46}$$

The main difficulty here is to find the explicit form for the matrix elements of $G^t(q)$ and $G^r(x)$. This problem was solved in our previous work [3]. We found that these matrices have the following structure:

$$\left(G^t(q)\right)_{i,j} = \delta_{i,j}\, q^{-\left(\frac{N}{2}\left(\frac{i^2}{N^2}-\frac{i}{n}\right)+\frac{N^2-1}{12N}\right)}. \tag{47}$$

The matrix for rational transforation $G^r(x)$ can be presented as a product:

$$G^r(x) = S_2\,S_1, \tag{48}$$

where S_2 is a diagonal matrix:

$$S_2 = \mathrm{diag}(x^{b_1}, x^{b_2}, \ldots, x^{b_{N-1}}, x^{b_N}),\quad b_i = -\frac{N(N-1)}{2N} + \frac{1-(i-1)}{N} \tag{49}$$

and S_2 is a constant matrix (*i.e.* is independent of x):

$$(S_2)_{i,j} = \varepsilon(i\neq N)\varepsilon(j\leq i)\frac{(i-1)!}{(j-i)!(j-1)!} + \varepsilon(i=N)\frac{N!}{j!(N-j)!}. \tag{50}$$

Here ε(condition) is equal to 1 if the condition is true and 0 otherwise. Knowing matrices of gauge transformations (47) and (48) we can easily use modern programs such as Maple or Mathematica and formulas (45) and (46) to compute the nonstandard trigonometric and rational R-matrices for $N \leq 15$. Here, as an example, we present the results of these calculations for N=3. For the $sl(3,\mathbb{C})$ non-standard trigonometric R-matrix we find:

$$\widetilde{R}^{trig}(u) = \begin{bmatrix} R^t_{1,1} & R^t_{1,2} & R^t_{1,3} \\ R^t_{2,1} & R^t_{2,2} & R^t_{2,3} \\ R^t_{3,1} & R^t_{3,2} & R^t_{3,3} \end{bmatrix}, \tag{51}$$

where for the matrix elements we have explicitly:

$$R^t_{1,1} = \begin{bmatrix} \sin(\pi(u+2\eta)) & 0 & 0 \\ 0 & \sin(\pi u)\,e^{2/3 i\pi\eta} & 0 \\ 0 & 0 & \sin(\pi u)\,e^{-2/3 i\pi\eta} \end{bmatrix},$$

$$R^t_{1,2} = \begin{bmatrix} 0 & 0 & 0 \\ \sin(2\pi\eta)\, e^{1/3 i\pi u} & 0 & 0 \\ 0 & 2\, i \sin(2\pi\eta)\sin(\pi u)\, e^{2/3 i\pi(-u-2\eta)} & 0 \end{bmatrix},$$

$$R^t_{1,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sin(2\,\pi\,\eta)\, e^{-1/3\, i\pi\, u} & 0 & 0 \end{bmatrix}, \qquad R^t_{2,1} = \begin{bmatrix} 0 & \sin(2\,\pi\,\eta)\, e^{-1/3\, i\pi\, u} & 0 \\ 0 & 0 & 0 \\ -2\, i \sin(2\,\pi\,\eta)\sin(\pi\, u)\, e^{2/3\, i\pi\,(u+2\,\eta)} & 0 & 0 \end{bmatrix},$$

$$R^t_{2,2} = \begin{bmatrix} \sin(\pi\, u)\, e^{-2/3\, i\pi\,\eta} & 0 & 0 \\ 0 & \sin(\pi\ (u+2\,\eta)) & 0 \\ 0 & 0 & \sin(\pi\, u)\, e^{2/3\, i\pi\,\eta} \end{bmatrix}, \qquad R^t_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sin(2\,\pi\,\eta)\, e^{1/3\, i\pi\, u} & 0 \end{bmatrix},$$

$$R^t_{3,1} = \begin{bmatrix} 0 & 0 & \sin(2\,\pi\,\eta)\, e^{1/3\, i\pi\, u} \\ 2\, i \sin(2\,\pi\,\eta)\sin(\pi\, u)\, e^{-2/3\, i\pi\,(u+2\,\eta)} & 0 & 0 \\ 0 & 4\, \sin(2\,\pi\,\eta)\sin(\pi\, u)\sin(\pi\ (u+2\,\eta))\, e^{1/3\, i\pi\,(u-2\,\eta)} & 0 \end{bmatrix},$$

$$R^t_{3,2} = \begin{bmatrix} 0 & -2\, i \sin(2\,\pi\,\eta)\sin(\pi\, u)\, e^{2/3\, i\pi\,(u+2\,\eta)} & 0 \\ 0 & 0 & \sin(2\,\pi\,\eta)\, e^{-1/3\, i\pi\, u} \\ 4\, \sin(2\,\pi\,\eta)\sin(\pi\, u)\sin(\pi\ (u+2\,\eta))\, e^{-1/3\, i\pi\,(u-2\,\eta)} & 0 & 0 \end{bmatrix},$$

$$R^t_{3,3} = \begin{bmatrix} \sin(\pi\, u)\, e^{2/3\, i\pi\,\eta} & 0 & 0 \\ 0 & \sin(\pi\, u)\, e^{-2/3\, i\pi\,\eta} & 0 \\ 0 & 0 & \sin(\pi\ (u+2\,\eta)) \end{bmatrix}.$$

The explicit expression for the non-standard trigonometric $sl(N,\mathbb{C})$ R-matrices was calculated in [12]. In the rational case we have the following R-matrix:

$$\widetilde{R}^{rat}(u) = \begin{bmatrix} R^r_{1,1} & R^r_{1,2} & R^r_{1,3} \\ R^r_{2,1} & R^r_{2,2} & R^r_{2,3} \\ R^r_{3,1} & R^r_{3,2} & R^r_{3,3} \end{bmatrix}$$

with the following matrix elements

$$R^r_{1,1} = \begin{bmatrix} 1/2\,\frac{u}{\eta}+1 & 0 & 0 \\ -2/3\, iu & 1/2\,\frac{u}{\eta} & 0 \\ -\frac{16}{27}\, iu^3 - \frac{64}{27}\, i\eta^2 u - \frac{16}{9}\, iu^2\eta & -4/3\, u^2 - 8/3\, u\eta & 1/2\,\frac{u}{\eta} \end{bmatrix}, \qquad R^r_{1,2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -4/3\, u^2 - 8/3\, u\eta & 2\, iu & 0 \end{bmatrix},$$

$$R^r_{1,3}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\qquad R^r_{2,1}=\begin{bmatrix}2/3\,iu&1&0\\-\frac{8}{9}\,u^2-\frac{16}{9}\,u\eta&0&0\\-\frac{32}{81}\,u^4-\frac{32}{27}\,u^3\eta-\frac{64}{27}\,u^2\eta^2-\frac{256}{81}\,u\eta^3&\frac{8}{27}\,iu^3+\frac{32}{9}\,iu\eta^2+\frac{16}{9}\,iu^2\eta&-2/3\,iu\end{bmatrix},$$

$$R^r_{2,2}=\begin{bmatrix}1/2\,\frac{u}{\eta}&0&0\\0&1/2\,\frac{u}{\eta}+1&0\\\frac{8}{9}\,iu^3+\frac{32}{27}\,i\eta^2u+\frac{16}{9}\,iu^2\eta&0&1/2\,\frac{u}{\eta}\end{bmatrix},\qquad R^r_{2,3}=\begin{bmatrix}0&0&0\\0&0&0\\-2/3\,iu&1&0\end{bmatrix},$$

$$R^r_{3,1}=\begin{bmatrix}\frac{16}{27}\,iu^3+\frac{64}{27}\,i\eta^2u+\frac{16}{9}\,iu^2\eta&-4/3\,u^2-8/3\,u\eta&1\\-\frac{32}{81}\,u^4-\frac{32}{27}\,u^3\eta-\frac{64}{27}\,u^2\eta^2-\frac{256}{81}\,u\eta^3&-\frac{8}{9}\,iu^3-\frac{16}{9}\,iu^2\eta-\frac{32}{27}\,i\eta^2u&2/3\,iu\\-\frac{1024}{243}\,u^2\eta^4-\frac{512}{243}\,u^3\eta^3-\frac{128}{243}\,u^5\eta-\frac{128}{729}\,u^6-\frac{256}{243}\,u^4\eta^2-\frac{4096}{729}\,u\eta^5&-\frac{32}{81}\,iu^5+\frac{512}{81}\,i\eta^4u-\frac{64}{81}\,iu^4\eta+\frac{256}{81}\,iu^2\eta^3&\frac{8}{27}\,iu^3-\frac{32}{27}\,i\eta^2u\end{bmatrix},$$

$$R^r_{3,2}=\begin{bmatrix}-4/3\,u^2-8/3\,u\eta&-2\,iu&0\\-\frac{8}{27}\,iu^3-\frac{32}{9}\,iu\eta^2-\frac{16}{9}\,iu^2\eta&0&1\\-\frac{512}{81}\,i\eta^4u-\frac{256}{81}\,iu^2\eta^3+\frac{64}{81}\,iu^4\eta+\frac{32}{81}\,iu^5&-\frac{32}{27}\,u^4-\frac{32}{9}\,u^3\eta-\frac{64}{9}\,u^2\eta^2-\frac{256}{27}\,u\eta^3&4/3\,u^2+8/3\,u\eta\end{bmatrix},$$

$$R^r_{3,3}=\begin{bmatrix}1/2\,\frac{u}{\eta}&0&0\\2/3\,iu&1/2\,\frac{u}{\eta}&0\\-\frac{8}{27}\,iu^3+\frac{32}{27}\,i\eta^2u&4/3\,u^2+8/3\,u\eta&1/2\,\frac{u}{\eta}+1\end{bmatrix}.$$

7. Conclusion

In this paper we demonstrated that there exist some new series of trigonometric and rational solutions of the Quantum Yang-Baxter equation for the algebras $sl(N)$. We have shown that these new R-matrices can be obtained from the elliptical Belavin-Drinfeld R-matrices by applying some singular gauge transformation to them. The singularity of the gauge transformation leads to the fact that the new matrices are not gauge equivalent to the known standard degenerations.

We revealed that new trigonometric and rational R-matrices in $sl(2)$ appear to be related with the standard degeneration by the quasi-Hopf twist. We conjecture that in the $sl(N)$ case the same twist should exist.

We derived the quantum algebras related to new R-matrices in the $sl(2)$ case and showed that these algebras can be considered as special limits of the Sklyanin algebra. By analogy with the work [4] we obtained a representation for these algebras in terms of the difference operators.

At the end of this paper we would like to add a few words about some problems. First of all it is interesting to find an explicit expression for the rational R-matrix in the $sl(N)$-case. The author tried to find the answer by calculating the matrix elements of the rational R-matrix for N=2,3,4... and then by guessing the full answer. For particular matrix elements it is not difficult to find the answer but for the full R-matrix the expressions seems to be very complicated. Probably, simpler way to derive this expression exist.

Another problem is to construct the representations for degenerate algebras in the $sl(N)$ case. In particular, it seems to be interesting to derive the dependence of the eigenvalues of the operator S_0 (which is gauge equivalent to the Hamiltonian of Ruijsenaars system) on different choices of finite dimensional representations.

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