

Effective electromagnetic lagrangian at finite temperature and density in the electroweak model

Research Article

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Abstract: Using the exact propagators in a constant magnetic field, the effective electromagnetic lagrangian at finite temperature and density is calculated to all orders in the field strength B within the framework of the complete electroweak model, in the weak coupling limit. The partition function and free energy are obtained explicitly and the finite temperature effective coupling is derived in closed form. Some implications of this result, potentially interesting to astrophysics and cosmology, are discussed.

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1. Introduction

Large magnetic fields are present in a variety of astrophysical sites like supernovae, neutron stars and white dwarfs, and even larger fields can arise in supernovae explosions or coalescing neutron stars. The remnants of such astrophysical cataclysms are called magnetars, young neutron stars with magnetic fields $10^{14} - 10^{16}$ G [1–3]. It has been argued that during the electroweak phase transition local magnetic fields much stronger than those of a magnetar could have existed, with field strength as high as $10^{22} - 10^{24}$ G [4–6]. In many of these situations the thermal and density effects of the medium are very important and must be considered, and many of the thermodynamical quantities that characterize the medium, like free energy, thermodynamical partition function and

effective potential are closely related to the effective action and effective lagrangian.

The work of Heisenberg and Euler [7], who calculated the one-loop vacuum effective lagrangian for spinor QED in a uniform electromagnetic background field, and that of Weisskopf [8], who calculated the analogous quantity for scalar QED, were published many years ago and are the first examples of what we now call low energy effective field theory. These pioneering papers lead to a number of important physical insights and applications: light-light scattering in QED [9], pair production from vacuum in the presence of an electric field [10–12] and vacuum birefringence [13], among others. The one-loop QED effective lagrangian at finite temperature and density has been investigated in magnetic field background [14–17], in electric field background [18, 19], in general background fields for the case of $0 + 1$ [20, 21] and $1 + 1$ dimensional massless QED [22, 23] and is very relevant and closely related to many physical phenomena such as, for example, the Casimir effect. When evaluating the QED

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effective lagrangian at finite temperature, the time component of the momentum four vector, over which we integrate, takes on only discrete values for a fixed temperature, while when computing the Casimir energy, an analogous substitution takes place in a space component of the momentum vector for a fixed distance between the plates. Recently there has been much interest [24–26] in studying the finite temperature Casimir effect in higher dimensional space-time models with compactified extra dimensions, like the Randall–Sundrum models [27, 28].

A literature search reveals that the effective lagrangian in a background magnetic field at finite temperature and density has been studied within the framework of QED [10–12, 18], or within the framework of the electroweak model but prior to the breaking of the electroweak symmetry, when the magnetic fields that are present belong to the $U(1)$ group of hypercharge and hence are called hypermagnetic fields [29–31]. The electroweak phase transition at finite temperature and in an external hypercharge magnetic field has been studied non-perturbatively using numerical techniques such as lattice Monte Carlo simulations [32], and perturbatively [33].

The finite temperature and density QED effective lagrangian for the complete electroweak model after symmetry breaking, however, has never been obtained analytically, for arbitrary values of the magnetic field. A recent paper by Navarro et al. [34], analytically derives the effective potential at finite temperature in a constant, though weak magnetic field. When magnetic fields are much larger than

$$B_e = \frac{m_e^2}{e} \simeq 4.414 \times 10^{13} \text{ G},$$

where m_e and $-e$ are the electron mass and charge respectively, the full electroweak model must be used, since electroweak magnetism [35] becomes important. In this paper, using Schwinger’s proper time method [10], I calculate the effective electromagnetic lagrangian of the complete electroweak model for a thermal environment treated exactly in the external constant magnetic field and in the weak coupling limit, i.e. with no virtual photons present.

In Section 2 the notation for the fermion, gauge boson and scalar thermal propagators in a constant magnetic field background is presented. In Section 3 the finite temperature and density effective electromagnetic lagrangian for the complete electroweak model is obtained. An extended discussion of several implications of my results is presented in Section 4. In the Appendix evaluate the next to leading order correction to the Heisenberg–Euler effective lagrangian and to the Weisskopf effective lagrangian of scalar QED in the case of strong magnetic field.

2. Vacuum and thermal propagators in a constant magnetic field

The metric used in this paper is $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and the z -axis points in the direction of the constant magnetic field \mathbf{B} . Therefore the electromagnetic field strength tensor $F^{\mu\nu}$ has only two nonvanishing components $F^{21} = -F^{12} = B$.

I start by recalling the expressions for the charged lepton $S_0(x', x'')$ [10, 36], W boson $G_0^{\mu\nu}(x', x'')$ and scalar vacuum propagators $D_0(x', x'')$ [40] in a constant magnetic field. These propagators are written in the Feynman gauge and derived using Schwinger’s proper time method:

$$S_0(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} S_0(k), \quad (1)$$

$$G_0^{\mu\nu}(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} G_0^{\mu\nu}(k), \quad (2)$$

$$D_0(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} D_0(k). \quad (3)$$

I choose the Landau gauge for the electromagnetic vector potential $A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu$ and therefore the phase factor which appears in Equations (1)–(3) is given by [36]

$$\Omega(x', x'') = \exp\left(-i \frac{e}{2} x'_\mu F^{\mu\nu} x''_\nu\right). \quad (4)$$

The translationally invariant parts of the propagators are

$$S_0(k) = \int_0^\infty \frac{ds}{\cos eBs} \exp\left[-is \left(m^2 - k_\parallel^2 - k_\perp^2 \frac{\tan eBs}{eBs} - i\epsilon\right)\right] \left[(m + \not{k}_\parallel) e^{-ieBs\sigma_3} + \frac{\not{k}_\perp}{\cos eBs} \right], \quad (5)$$

$$G_0^{\mu\nu}(k) = - \int_0^\infty \frac{ds}{\cos eBs} \exp\left[-is \left(M^2 - k_\parallel^2 - k_\perp^2 \frac{\tan eBs}{eBs} - i\epsilon\right)\right] \left[g_\parallel^{\mu\nu} - (e^{2eFs})^{\mu\nu} \right], \quad (6)$$

$$D_0(k) = \int_0^\infty \frac{ds}{\cos eBs} \exp\left[-is \left(M^2 - k_\parallel^2 - k_\perp^2 \frac{\tan eBs}{eBs} - i\epsilon\right)\right], \quad (7)$$

where $-e$ and m are the charge and mass of the charged lepton, M is the W -mass, and the $-i\epsilon$ prescription for the propagators is essential for the convergence of the s integrals. It is convenient to use the notation

$$a_{\parallel}^{\mu} = (a^0, 0, 0, a^3), \quad a_{\perp}^{\mu} = (0, a^1, a^2, 0) \quad (8)$$

and

$$(ab)_{\parallel} = a^0 b^0 - a^3 b^3, \quad (ab)_{\perp} = -a^1 b^1 - a^2 b^2 \quad (9)$$

for arbitrary four-vectors a and b . Using this notation I write the metric tensor as

$$g^{\mu\nu} = g_{\parallel}^{\mu\nu} + g_{\perp}^{\mu\nu}. \quad (10)$$

The 4×4 matrix σ_3 that appears in the charged lepton propagator (5), is given by

$$\sigma_3 = \frac{i}{2} [\gamma^1, \gamma^2]. \quad (11)$$

When writing the W propagator (6), I use the notation

$$(e^{2eFs})^{\mu\nu} = -g_{\perp}^{\mu\nu} \cos(2eBs) + \frac{F^{\mu\nu}}{B} \sin(2eBs). \quad (12)$$

Notice that, as $B \rightarrow 0$, the phase factor $\Omega \rightarrow 1$ and the translationally invariant part of the propagators take the standard form found in many quantum field theory textbooks

$$\begin{aligned} S_0(k) &\rightarrow i \frac{k + m}{k^2 - m^2 + i\epsilon}, \\ G_0^{\mu\nu}(k) &\rightarrow \frac{-ig^{\mu\nu}}{k^2 - M^2 + i\epsilon}, \\ D_0(k) &\rightarrow \frac{i}{k^2 - M^2 + i\epsilon}. \end{aligned} \quad (13)$$

At finite temperature and density we need to replace the vacuum propagators in Equations (5)-(7) with thermal propagators. There are three distinct formalisms to describe finite temperature field theories [37], the imaginary time formalism [38, 39], and two different real time formalism, one known as thermo field dynamics (TFD) and the other known as closed time path formalism. Following the lead of Ref. [15], where the TFD propagator for Dirac fermion fields in the presence of a constant magnetic field is obtained, I will use the real time formalism of thermo field dynamics, which is suitable for a plasma in thermal equilibrium. While the imaginary time formalism trades the time variable for the temperature, the real time

formalism of TFD retains the time coordinate in addition to the temperature by doubling the Hilbert space of the theory. This doubling of the field degrees of freedom is achieved by introducing thermal "ghost" fields. Of course only the "physical" fields occur on external lines of the Green's functions. Each field and its thermal "ghost" are therefore grouped into thermal doublets, and the thermal propagators become 2×2 matrices with off-diagonal elements. The $1 - 1$ components of the TFD propagators $S(x', x'')$, $G^{\mu\nu}(x', x'')$ and $D(x', x'')$ are easily constructed starting from the proper-time form of the vacuum propagators

$$S(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} S(k), \quad (14)$$

$$G^{\mu\nu}(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} G^{\mu\nu}(k), \quad (15)$$

$$D(x', x'') = \Omega(x', x'') \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x'')} D(k), \quad (16)$$

where $S(k)$, $G^{\mu\nu}(k)$ and $D(k)$ are defined in terms of the translationally invariant parts of the vacuum propagators, $S_0(k)$, $G_0^{\mu\nu}(k)$ and $D_0(k)$, and of the fermion and boson occupation numbers $f_F(k^0)$ and $f_B(k^0)$

$$S(k) = S_0(k) - f_F(k^0) [S_0(k) + S_0^*(k)], \quad (17)$$

$$G^{\mu\nu}(k) = G_0^{\mu\nu}(k) + f_B(k^0) [G_0^{\mu\nu}(k) + G_0^{\mu\nu*}(k)], \quad (18)$$

$$\Delta(k) = \Delta_0(k) + f_B(k^0) [\Delta_0(k) + \Delta_0^*(k)]. \quad (19)$$

Notice that in Equations (17)-(19) the pieces proportional to the occupation numbers represent the thermal parts of the propagators. The fermion occupation number at temperature T and chemical potential μ is defined as

$$f_F(k^0) = f_F^+(k^0) \theta(k^0) + f_F^-(k^0) \theta(-k^0) \quad (20)$$

with

$$f_F^{\pm}(k^0) = \frac{1}{e^{\pm\beta(k^0 - \mu)} + 1}, \quad (21)$$

and the boson occupation number $f_B(k^0)$ is defined as

$$f_B(k^0) = \frac{1}{e^{\beta|k^0|} - 1}, \quad (22)$$

with $\beta = T^{-1}$. These thermal propagators of the "physical" fields cannot be used naively for perturbative expansions, but in most perturbative calculations one needs to include thermal "ghost" fields. There is a delta function hidden in the thermal part of the propagators, and

the overlap of delta functions with coinciding arguments on several internal legs leads to expressions that are not well defined.

I will indicate the matrix propagators with a tilde, and below I will write the full matrix form of the translationally invariant part of the fermion, charged vector and scalar field thermal propagator. The fermion propagator in TFD is

$$\tilde{S}(k) = U_F(k^0) \begin{pmatrix} S_0(k) & 0 \\ 0 & -S_0^*(k) \end{pmatrix} U_F^T(k^0), \quad (23)$$

where

$$U_F(k^0) = \begin{pmatrix} \cos \vartheta_F(k^0) & -\sin \vartheta_F(k^0) \\ \sin \vartheta_F(k^0) & \cos \vartheta_F(k^0) \end{pmatrix} \quad (24)$$

and

$$\sin \vartheta_F(k^0) = \theta(k^0) \sqrt{f_F^+(k^0)} - \theta(-k^0) \sqrt{f_F^+(-k^0)}, \quad (25)$$

$$\cos \vartheta_F(k^0) = \theta(k^0) \sqrt{1 - f_F^+(k^0)} - \theta(-k^0) \sqrt{1 - f_F^+(-k^0)}. \quad (26)$$

Here, $U_F^T(k^0)$ is the transpose of $U_F(k^0)$. The $\tilde{S}(k)_{11}$ component is, of course, the same as (17) and the other components are only used in higher loop calculations. The matrix forms of the translationally invariant part of the charged vector and scalar field thermal propagators are

$$\tilde{G}^{\mu\nu}(k) = U_B(k^0) \begin{pmatrix} C_0^{\mu\nu}(k) & 0 \\ 0 & C_0^{\mu\nu*}(k) \end{pmatrix} U_B(k^0), \quad (27)$$

$$\tilde{D}(k) = U_B(k^0) \begin{pmatrix} D_0(k) & 0 \\ 0 & D_0^*(k) \end{pmatrix} U_B(k^0), \quad (28)$$

with

$$U_B(k^0) = \begin{pmatrix} \sqrt{1 + f_B(k^0)} & \sqrt{f_B(k^0)} \\ \sqrt{f_B(k^0)} & \sqrt{1 + f_B(k^0)} \end{pmatrix}. \quad (29)$$

The $\tilde{G}^{\mu\nu}(k)_{11}$ and $\tilde{D}(k)_{11}$ components are the same as (18) and (19) respectively, and the other components are only needed in higher loop calculations.

3. Finite temperature and density effective electromagnetic lagrangian

In vacuum the effective EM action S_{eff} of the electroweak model at the one-loop level can be written as

$$S_{eff} = \int d^4x \mathcal{L}_{eff} \quad (30)$$

and the effective EM lagrangian \mathcal{L}_{eff} is given by

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}, \quad (31)$$

where the tree level part in the case of a constant magnetic field is the classical lagrangian density

$$\mathcal{L}^{(0)} = -\frac{1}{2}B^2 \quad (32)$$

and $\mathcal{L}^{(1)}$ is the one-loop quantum correction. In the Feynman gauge, all the fields of the electroweak model contribute to the one-loop effective lagrangian

$$\mathcal{L}^{(1)} = \sum_f \mathcal{L}_f + \mathcal{L}_W + \sum_s \mathcal{L}_s + \mathcal{L}_\nu + \mathcal{L}_N, \quad (33)$$

where \mathcal{L}_f , \mathcal{L}_s , \mathcal{L}_W indicate the contributions of charged fermion, charged scalar and W -fields respectively, and we need to sum over the quarks and charged leptons and over the non-physical charged scalars. \mathcal{L}_N and \mathcal{L}_ν indicate the contribution of the neutral (gauge and scalar) bosons, and the neutrino contribution. While the neutral fermion, gauge and scalar boson fields do not feel the effects of the magnetic field, their contribution needs to be added to the effective lagrangian to the same order of the approximation, in order to account for all the electroweak model degrees of freedom.

In a medium at finite temperature and density, we still use Equation (33) to calculate $\mathcal{L}^{(1)}$, but the contributions of the various fields are evaluated using thermal propagators instead of vacuum propagators, and we find that $\mathcal{L}^{(1)}$ is the sum of a vacuum part \mathcal{L}^0 and a thermal part \mathcal{L}^T

$$\mathcal{L}^{(1)} = \mathcal{L}^0 + \mathcal{L}^T. \quad (34)$$

The same is true for $\mathcal{L}_f = \mathcal{L}_f^0 + \mathcal{L}_f^T$, for \mathcal{L}_W and for \mathcal{L}_s . \mathcal{L}_N and \mathcal{L}_ν have a vanishing vacuum part, since the neutral fields do not interact with the magnetic field, and will only have a thermal part that does not depend on B . Summarizing, $\mathcal{L}^0 = \sum_f \mathcal{L}_f^0 + \mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0$ and $\mathcal{L}^T = \sum_f \mathcal{L}_f^T + \mathcal{L}_W^T + \sum_s \mathcal{L}_s^T + \mathcal{L}_N^T + \mathcal{L}_\nu^T$.

At the one-loop level thermal "ghost" fields do not contribute to the finite temperature effective lagrangian, and therefore I will only use the $1-1$ components of the thermal propagators, as listed in Equations (17)-(19). \mathcal{L}_f was calculated in Refs. [14, 15] using the solutions to the Dirac equation in a uniform magnetic field to construct the fermion thermal propagator. In this paper I calculate \mathcal{L}_f using the thermal propagator of Equation (14), which is constructed using the Schwinger proper time method, and use the following identity [14, 15] to evaluate the contribution to the effective lagrangian of a charged lepton field whose thermal propagator is $S(x', x'')$

$$\frac{\partial \mathcal{L}_f}{\partial m} = \text{Tr} S(x, x), \quad (35)$$

where the trace is over spinor indices. Eq. (35), and the analogous Eqs. (55) and (66) that will be used later when evaluating \mathcal{L}_W and \mathcal{L}_s , are not valid beyond the one loop order. Identical, or similar relations are not available when one calculates the effective Lagrangian at ring order by computing ring diagrams, consisting in the resummation of successive insertions of self-energies in vacuum bubbles. Calculation of ring order diagrams is necessary when studying the electroweak phase transition. The generalization of Eq. (35) to quarks is straightforward. After taking the spinor trace, I find

$$\frac{\partial \mathcal{L}_f^0}{\partial m} = 4m \int \frac{d^4 k}{(2\pi)^4} \int_0^\infty ds \mathcal{E}(m) \quad (36)$$

and

$$\frac{\partial \mathcal{L}_f^T}{\partial m} = -4m \int \frac{d^4 k}{(2\pi)^4} f_F(k^0) \int_0^\infty ds [\mathcal{E}(m) + \mathcal{E}^*(m)], \quad (37)$$

where I introduce the notation

$$\mathcal{E}(m) = \exp \left[-is \left(m^2 - k_{\parallel}^2 - k_{\perp}^2 \frac{\tan eBs}{eBs} - i\epsilon \right) \right]. \quad (38)$$

After the straightforward integration over the four variables k^μ and a rotation of the s contour to the negative imaginary axis, I obtain immediately the derivative of the vacuum part

$$\frac{\partial \mathcal{L}_f^0}{\partial m} = \frac{m}{4\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-sm^2} eBs \coth(eBs), \quad (39)$$

and find

$$\mathcal{L}_f^0 = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left[eBs \coth(eBs) - 1 - \frac{(eBs)^2}{3} \right], \quad (40)$$

which has been renormalized by adding a second order polynomial in eB and reproduces the old result by Schwinger [10]. This charge and wave function renormalization procedure leaves eB invariant and produces a contribution of $\frac{2\alpha}{3\pi}$ to the QED β function from our fermion field.

In order to obtain the thermal part \mathcal{L}_f^T , I start by integrating over the three k^i variables to find

$$\frac{\partial \mathcal{L}_f^T}{\partial m} = -\frac{m}{2\pi^{\frac{3}{2}}} \int_{-\infty}^{+\infty} dk^0 f_F(k^0) \text{Re} \int_0^\infty ds \mathcal{I}_f(s), \quad (41)$$

where

$$\mathcal{I}_f(s) = \frac{e^{-is(m^2 - k_0^2 - i\epsilon)}}{(is)^{\frac{3}{2}}} eBs \cot(eBs). \quad (42)$$

Next I need to rotate the s integration contour to the positive imaginary axis, and must be careful with its convergence and analytic structure. After the contour rotation, I find only contribution from $|k^0| > m$, as expected, and, since the rotated contour crossed the poles of $\mathcal{I}(s)$, I need to include a sum over the residues of $\mathcal{I}(s)$ at its poles $s = \frac{n\pi}{eB}$ (for $n \geq 1$)

$$\begin{aligned} 2\pi i \text{Res} \left[\mathcal{I}_f \left(s = \frac{n\pi}{eB} \right) \right] \\ = 2 \left(\frac{\pi eB}{n} \right)^{\frac{1}{2}} \exp \left[\frac{in\pi}{eB} (k_0^2 - m^2) - \frac{i\pi}{4} \right]. \end{aligned} \quad (43)$$

I obtain the following s -integral and sum over the real parts of the residues of $\mathcal{I}(s)$

$$\begin{aligned} \frac{\partial \mathcal{L}_f^T}{\partial m} = & \frac{m}{2\pi^{\frac{3}{2}}} \int_{-\infty}^{+\infty} dk^0 f_F(k^0) \theta(k_0^2 - m^2) \\ & \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}} e^{-s(k_0^2 - m^2)} eBs \coth(eBs) \right. \\ & \left. - 2 \sum_{n=1}^{\infty} \left(\frac{\pi eB}{n} \right)^{\frac{1}{2}} \cos \left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - m^2) \right) \right] \end{aligned} \quad (44)$$

and find the fermion contribution to the effective Lagrangian

$$\begin{aligned} \mathcal{L}_f^T = & \frac{1}{4\pi^{\frac{3}{2}}} \int_{-\infty}^{+\infty} dk^0 f_F(k^0) \theta(k_0^2 - m^2) \\ & \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}} e^{-s(k_0^2 - m^2)} eBs \coth(eBs) \right. \\ & \left. - \frac{2}{\pi^{\frac{1}{2}}} \sum_{n=1}^{\infty} \left(\frac{eB}{n} \right)^{\frac{3}{2}} \sin \left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - m^2) \right) \right], \end{aligned} \quad (45)$$

which agrees with the result obtained by a different method in Refs. [14, 15]. Using the form of \mathcal{L}_f^T given in Equation (45), it is shown in Ref. [14] that a fermion-antifermion plasma in a magnetic field exhibits de Haas-van Alphen oscillations in the limit where $T = 0$, and $eB \ll \mu^2 - m^2 \ll m^2$. The second term in Equation (45) is responsible for the oscillatory behavior, and the oscillation frequency agrees with the one derived by Onsager for the de Haas-van Alphen effect. We can write \mathcal{L}_f^T in another form that seems less transparent of its physical content but clearly displays its connection to the partition function $Z_f(B, T, \mu)$ of a relativistic fermion-antifermion gas in a magnetic field in a sufficiently large quantization volume V . Starting from Equation (37), I integrate over the k_\perp variables and do a Wick rotation of the s variable to obtain

$$\frac{\partial \mathcal{L}_f^T}{\partial m} = \frac{m}{2\pi^3} \int d^2 k_\parallel f_F(k^0) \operatorname{Re} \int_0^\infty ds \exp[-s(m^2 - k_\parallel^2 - i\epsilon)] ieB \coth(eBs). \quad (46)$$

Next I use the following identity

$$\coth(eBs) = \sum_{n=0}^{\infty} \sum_{\lambda=1}^2 e^{-2eBs(n+\lambda-1)}, \quad (47)$$

and do the s integration, then use

$$\operatorname{Re} \frac{i}{x - i\epsilon} = -\pi \delta(x), \quad (48)$$

and finally integrate over k^0 to obtain

$$\frac{\partial \mathcal{L}_f^T}{\partial m} = -m \frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=1}^2 \int_{-\infty}^{\infty} dk \frac{f_F(\omega_{n,\lambda}) + f_F(-\omega_{n,\lambda})}{\omega_{n,\lambda}}, \quad (49)$$

where

$$\omega_{n,\lambda} = \sqrt{m^2 + k^2 + 2eB(n + \lambda - 1)} \quad (50)$$

are the standard Landau energy levels for a spin- $\frac{1}{2}$ fermion and λ is the spin quantum number. \mathcal{L}_f^T is now obtained immediately with an integration by parts with respect to k

$$\mathcal{L}_f^T = \frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda=1}^2 \int_{-\infty}^{\infty} dk \frac{k^2}{\omega_{n,\lambda}} [f_F(\omega_{n,\lambda}) + f_F(-\omega_{n,\lambda})]. \quad (51)$$

With \mathcal{L}_f^T written in this form, its connection to the partition function of a relativistic fermion-antifermion plasma

placed in a box of volume V and in the presence of an external magnetic field B is apparent [14–16]

$$\mathcal{L}_f^T = \frac{\ln Z_f(B, T, \mu)}{\beta V} \quad (52)$$

and so is \mathcal{L}_f^T connection to the free energy $F_f(B, T, \mu)$ of the fermion-antifermion plasma

$$\mathcal{L}_f^T = \frac{F_f(B, T, \mu)}{V}. \quad (53)$$

The strong field limit is more easily obtained using Equation (51). For $eB \gg T^2, m^2, \mu^2$ only the lowest Landau level contributes to \mathcal{L}_f^T , and we find

$$\mathcal{L}_f^T = \frac{eB}{4\pi^2} \int_{-\infty}^{\infty} dk \frac{k^2}{\omega_{0,1}} [f_F(\omega_{0,1}) + f_F(-\omega_{0,1})], \quad (54)$$

where the energy of the lowest Landau level is $\omega_{0,1} = \sqrt{k^2 + m^2}$. It is shown in Ref. [15], through a rather lengthy calculation, that also Equation (45) yields the same result. A complete analysis of \mathcal{L}_f^T for various large or small limits of the dimensional parameters T, B, m and μ , will be done in Sec. 4.

The contribution of the W fields to the effective lagrangian is obtained using the following

$$\frac{\partial \mathcal{L}_W}{\partial M^2} = G_\mu^\mu(x, x), \quad (55)$$

where $G^{\mu\nu}(x', x'')$ is the thermal propagator of Equation (15). After taking the trace, the derivatives of the vacuum and thermal part of \mathcal{L}_W are found to be

$$\frac{\partial \mathcal{L}_W^0}{\partial M^2} = - \int \frac{d^4 k}{(2\pi)^4} \int_0^\infty ds \mathcal{E}(M) \frac{2 + 2 \cos(2eBs)}{\cos(eBs)} \quad (56)$$

and

$$\frac{\partial \mathcal{L}_W^T}{\partial M^2} = - \int \frac{d^4 k}{(2\pi)^4} f_B(k^0) \int_0^\infty ds [\mathcal{E}(M) + \mathcal{E}^*(M)] \frac{2 + 2 \cos(2eBs)}{\cos(eBs)}, \quad (57)$$

where \mathcal{E} is defined in Equation (38). I do the k^μ integration and a Wick rotation of the s variable to obtain

$$\frac{\partial \mathcal{L}_W^0}{\partial M^2} = - \frac{1}{(2\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-sM^2} eBs \coth(eBs) \cosh(eBs) \quad (58)$$

and find the vacuum contribution of the W fields to the effective lagrangian [40]

$$\mathcal{L}_W^0 = \frac{1}{(2\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-sM^2} \left[eBs \coth(eBs) \cosh(eBs) - 1 - \frac{5}{6}(eBs)^2 \right], \quad (59)$$

renormalized by adding a second order polynomial in eB . I now use Equation (57) and integrate over the k^i variables to find

$$\frac{\partial \mathcal{L}_W^T}{\partial M^2} = -\frac{1}{4\pi^{\frac{5}{2}}} \int_{-\infty}^{+\infty} dk^0 f_B(k^0) \operatorname{Re} \int_0^\infty ds \mathcal{I}_W(s), \quad (60)$$

where

$$\mathcal{I}_W(s) = \frac{e^{-is(M^2 - k_0^2 - i\epsilon)}}{(is)^{\frac{3}{2}}} \frac{eBs}{\sin(eBs)} [1 + \cos(2eBs)]. \quad (61)$$

Next I deform the s integration contour to the imaginary axis and cross the poles of $\mathcal{I}_W(s)$ at $s = \frac{n\pi}{eB}$ in the process. After including the contributions of the residues of these poles, I obtain

$$\begin{aligned} \frac{\partial \mathcal{L}_W^T}{\partial M^2} = & \frac{1}{8\pi^{\frac{5}{2}}} \sum_{\lambda'=-1}^{+1} (2 - |\lambda'|) \int_{-\infty}^{+\infty} dk^0 f_B(k^0) \theta[k_0^2 - M^2 + (2|\lambda'| - 1)eB] \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}} e^{-s(k_0^2 - M^2)} \right. \\ & \left. \times e^{-2\lambda' eBs} \frac{eBs}{\sinh(eBs)} - 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi eB}{n} \right)^{\frac{1}{2}} \cos\left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - M^2)\right) \right], \end{aligned} \quad (62)$$

and find

$$\begin{aligned} \mathcal{L}_W^T = & \frac{1}{8\pi^{\frac{5}{2}}} \sum_{\lambda'=-1}^{+1} (2 - |\lambda'|) \int_{-\infty}^{+\infty} dk^0 f_B(k^0) \theta[k_0^2 - M^2 + (2|\lambda'| - 1)eB] \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}} e^{-s(k_0^2 - M^2)} \right. \\ & \left. \times e^{-2\lambda' eBs} \frac{eBs}{\sinh(eBs)} - \frac{2}{\pi^{\frac{1}{2}}} \sum_{n=1}^{\infty} (-1)^n \left(\frac{eB}{n} \right)^{\frac{3}{2}} \sin\left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - M^2)\right) \right]. \end{aligned} \quad (63)$$

The W contribution to the effective electromagnetic lagrangian at finite temperature is one of the main results of this paper. \mathcal{L}_W^T can be written in a different form, that is obtained using steps similar to those outlined in Equations (46)-(51)

$$\mathcal{L}_W^T = \frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda'=-1}^{+1} (2 - |\lambda'|) \int_{-\infty}^{\infty} dk \frac{k^2}{E_{n,\lambda'}} [f_B(E_{n,\lambda'}) + f_B(-E_{n,\lambda'})], \quad (64)$$

where the spin quantum number λ' takes the three values $0, \pm 1$ and the Landau energy levels for a spin-1 boson are

$$E_{n,\lambda'} = \sqrt{M^2 + k^2 + (2n + 2\lambda' + 1)eB}. \quad (65)$$

Last I obtain the scalar field contribution to the effective lagrangian using the following

$$\frac{\partial \mathcal{L}_s}{\partial M^2} = -D(x, x), \quad (66)$$

where $D(x', x'')$ is the scalar propagator. The vacuum part of \mathcal{L}_s is [8, 40]

$$\mathcal{L}_s^0 = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-sM^2} \left[\frac{eBs}{\sinh(eBs)} - 1 + \frac{1}{6}(eBs)^2 \right], \quad (67)$$

renormalized by adding a second order polynomial in eB . The thermal part of the scalar field contribution is [15]

$$\mathcal{L}_s^T = \frac{1}{8\pi^{\frac{5}{2}}} \int_{-\infty}^{+\infty} dk^0 f_B(k^0) \theta(k_0^2 - M^2 - eB) \left[\int_0^\infty \frac{ds}{s^{\frac{5}{2}}} e^{-s(k_0^2 - M^2)} \right. \\ \left. \times \frac{eBs}{\sinh(eBs)} - \frac{2}{\pi^{\frac{1}{2}}} \sum_{n=1}^{\infty} (-1)^n \left(\frac{eB}{n} \right)^{\frac{3}{2}} \sin\left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - M^2)\right) \right] \quad (68)$$

and I can write it also as

$$\mathcal{L}_s^T = \frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk \frac{k^2}{E_{n,0}} [f_B(E_{n,0}) + f_B(-E_{n,0})], \quad (69)$$

where $E_{n,0}$ are the Landau energy levels of Equation (65) with $\lambda' = 0$ for the spin-0 scalar field. Equation (69) shows the connection of \mathcal{L}_s^T to the partition function $Z_s(B, T)$ and free energy $F_s(B, T)$ of a spin-0 boson plasma placed in a volume V and in the presence of a constant magnetic

field B

$$\mathcal{L}_s^T = \frac{\ln Z_s(B, T)}{\beta V} = \frac{F_s(B, T)}{V}. \quad (70)$$

In the Feynman gauge the quantity $\sum \mathcal{L}_s$, which is the contribution of the non-physical charged scalars to the finite temperature effective lagrangian $\mathcal{L}^{(1)}$ at one loop, is obtained by adding the vacuum and thermal contributions coming from the Goldstone (scalar boson) field of mass M , and the two ghost fields (scalar fermions of mass M). The contribution of each of the ghost fields is opposite to that of the Goldstone boson, and therefore I find

$$\mathcal{L}_W^T + \sum_s \mathcal{L}_s^T = \frac{1}{8\pi^{\frac{5}{2}}} \sum_{\lambda'=-1}^{+1} \int_{-\infty}^{+\infty} dk^0 f_B(k^0) \theta[k_0^2 - M^2 + (2|\lambda'| - 1)eB] \left[\int_0^\infty \frac{ds}{s^{\frac{5}{2}}} e^{-s(k_0^2 - M^2)} \right. \\ \left. \times e^{-2\lambda'eBs} \frac{eBs}{\sinh(eBs)} - \frac{2}{\pi^{\frac{1}{2}}} \sum_{n=1}^{\infty} (-1)^n \left(\frac{eB}{n} \right)^{\frac{3}{2}} \sin\left(\frac{\pi}{4} - \frac{n\pi}{eB} (k_0^2 - M^2)\right) \right], \quad (71)$$

which can be equivalently written as

$$\mathcal{L}_W^T + \sum_s \mathcal{L}_s^T = \frac{eB}{4\pi^2} \sum_{n=0}^{\infty} \sum_{\lambda'=-1}^{+1} \int_{-\infty}^{\infty} dk \frac{k^2}{E_{n,\lambda'}} [f_B(E_{n,\lambda'}) \\ + f_B(-E_{n,\lambda'})], \quad (72)$$

where the Landau energy levels $E_{n,\lambda'}$ are given by Equation (65). The partition function $Z_W(B, T)$ and free energy $F_W(B, T)$ of a W boson plasma placed in a volume V in a constant magnetic field B is obtained immediately

$$\frac{\ln Z_W(B, T)}{\beta V} = \frac{F_W(B, T)}{V} = \mathcal{L}_W^T + \sum_s \mathcal{L}_s^T. \quad (73)$$

In the unitary gauge only physical particles are present, and the W contributes to the effective lagrangian with one

longitudinal mode and two transverse modes [40]. The charged unphysical scalars (Goldstone boson and ghost fields) have infinite mass and therefore are not present in the unitary gauge. In the Feynman gauge however, the W contributes to the effective lagrangian with two longitudinal modes and two transverse modes, and the contributions of the scalars to the effective lagrangian exactly cancel the contribution of one of the W longitudinal modes [40]. Since the W does not have a chemical potential despite its charge, and since the unphysical charged scalars are closely tied to the W in the way I described above, the scalars cannot have a chemical potential, because giving the scalars a chemical potential would be equivalent to assigning a chemical potential to only one of the W longitudinal modes.

The lepton and quark fields contribution is

$$\begin{aligned} \sum_f \mathcal{L}_f^T = & \sum_f \frac{N_f}{4\pi^{\frac{3}{2}}} \int_{-\infty}^{+\infty} dk^0 f_F(k^0) \theta(k_0^2 - m_f^2) \left[\int_0^\infty \frac{ds}{s^{\frac{3}{2}}} e^{-s(k_0^2 - m_f^2)} e q_f B s \coth(e q_f B s) \right. \\ & \left. - \frac{2}{\pi^{\frac{1}{2}}} \sum_{n=1}^\infty \left(\frac{e|q_f|B}{n} \right)^{\frac{3}{2}} \sin\left(\frac{\pi}{4} - \frac{n\pi}{e|q_f|B} (k_0^2 - m_f^2)\right) \right], \end{aligned} \quad (74)$$

where m_f is the fermion mass. The summation index f runs over the charged lepton fields e, μ, τ and the quark fields u, d, c, s, t, b and $N_f = 1$ and $q_f = 1$ for each of the three leptons and, taking into account the quark charges and colors, $N_f = 3$ and $q_f = -\frac{2}{3}$ or $q_f = \frac{1}{3}$ for the six quarks. Equation (74) can also be written as

$$\begin{aligned} \sum_f \mathcal{L}_f^T = & \sum_f N_f \frac{e|q_f|B}{4\pi^2} \sum_{n=0}^\infty \sum_{\lambda=1}^2 s \int_{-\infty}^\infty dk \frac{k^2}{\omega_{n,\lambda}} [f_F(\omega_{n,\lambda}) \\ & + f_F(-\omega_{n,\lambda})], \end{aligned} \quad (75)$$

where the Landau energy levels for a fermion are

$$\omega_{n,\lambda} = \sqrt{m_f^2 + k^2 + 2e|q_f|B(n + \lambda - 1)}. \quad (76)$$

Finally I present the thermal contribution of the neutral fields of the electroweak model $\mathcal{L}_\nu + \mathcal{L}_N$. These fields do not interact with the magnetic field and will not contribute a vacuum part to $\mathcal{L}^{(1)}$. The neutral gauge and scalar contribution is $\mathcal{L}_N = \mathcal{L}_\nu^T + \mathcal{L}_Z^T + \mathcal{L}_H^T$, where the photon contribution is

$$\mathcal{L}_\nu^T = \frac{T^4 \pi^2}{45}, \quad (77)$$

the sum of the contributions of the Z and the neutral Goldstone and ghost fields is

$$\mathcal{L}_Z^T = \frac{1}{2\pi^2} \int_0^\infty \frac{k^4 f_B(\omega_Z)}{\omega_Z} dk \quad (78)$$

and the contribution of the Higgs is

$$\mathcal{L}_H^T = \frac{1}{6\pi^2} \int_0^\infty \frac{k^4 f_B(\omega_H)}{\omega_H} dk, \quad (79)$$

with $\omega_Z = \sqrt{k^2 + M_Z^2}$ and $\omega_H = \sqrt{k^2 + M_H^2}$. M_Z and M_H are the Z -boson and Higgs mass respectively. The contribution of each of the three neutrino fields is

$$\mathcal{L}_\nu^T = \left(\frac{7}{8}\right) \frac{T^4 \pi^2}{45}, \quad (80)$$

where I neglect the small neutrino mass.

The fermionic contribution to the vacuum part of $\mathcal{L}^{(1)}$ is [7]

$$\begin{aligned} \sum_f \mathcal{L}_f^0 = & - \sum_f \frac{N_f}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-sm_f^2} \left[e q_f B s \coth(e q_f B s) \right. \\ & \left. - 1 - \frac{(e q_f B s)^2}{3} \right] \end{aligned} \quad (81)$$

and the W and scalar contribution is [40]

$$\begin{aligned} \mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0 = & \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left[\frac{eBs}{\sinh(eBs)} (1 + e^{2eBs}) \right. \\ & \left. + e^{-2eBs} - 3 - \frac{7}{2}(eBs)^2 \right]. \end{aligned} \quad (82)$$

The renormalization procedure that was used to obtain $\sum_f \mathcal{L}_f^0$ and $\mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0$ leads to the following value of the QED β function for the electroweak theory

$$\beta^{EW} = \frac{2\alpha}{3\pi} \sum_f N_f q_f^2 - \frac{7\alpha}{2\pi}, \quad (83)$$

where the summation is the contribution to β^{EW} from the fermions of the theory and the second term is the contribution from the W and the scalar fields.

4. Discussion and conclusions

The effective electromagnetic lagrangian at finite temperature and density $\mathcal{L}^{(1)}$ obtained in the previous section contains several dimensionful parameters T, B , the charged fermion and gauge boson masses and the fermion chemical potentials. Some of these parameters can be large or small compared to each other, and I will discuss some of these limits that I find most interesting.

First I analyze the strong field limit and investigate the case of $eB \gg m_e^2$, which is the natural scale for the

magnetic field strength required to significantly influence quantum processes. Magnetic fields of this magnitude are present in many astrophysical sites, and fields as large as

$$B_e = \frac{m_e^2}{e}$$

or larger can arise in magnetars, among others. In the literature [14, 15] the thermal and density corrections to the effective lagrangian have been neglected when compared to \mathcal{L}_f^0 for $f = e$ which, in the strong field limit and to leading order is

$$\mathcal{L}_e^0 \simeq \frac{(eB)^2}{24\pi^2} \ln \left(\frac{eB}{m_e^2} \right). \quad (84)$$

However a more accurate calculation of \mathcal{L}_f^0 presented in the Appendix, where the next to leading order term is derived, shows that

$$\mathcal{L}_e^0 = \frac{(eB)^2}{24\pi^2} \left[\ln \left(\frac{eB}{6m_e^2} \right) - \frac{1}{2} \right] \quad (85)$$

where γ_E is the Euler-Mascheroni constant. The maximum error introduced by using (88) is less than one percent. A numerical evaluation of \mathcal{L}_e^T using Equations (87) and (88) shows that, for a CP -symmetric plasma with $B \sim 10B_e$, the thermal correction given by Equation (87) is larger than the vacuum part of the effective lagrangian of Equation (86) for $T \geq \frac{m_e}{6}$ and, for $T = m_e$, I find that the thermal correction dominates with $\mathcal{L}_e^T \sim 50\mathcal{L}_e^0$.

It is also interesting to investigate the case of a finite density medium where μ_e is the electron chemical potential. For a finite density medium with strong magnetic field $eB \gg m_e^2 \sim \mu_e^2 - m_e^2 \gg T^2$, the electron occupation number is $f_F(\omega_e) \simeq \theta(\omega_e)\theta(\omega_e - \mu_e)$ and I obtain from Equation (54)

$$\mathcal{L}_e^T = \frac{eB}{4\pi^2} \left[\mu_e \sqrt{\mu_e^2 - m_e^2} - m_e^2 \ln \left(\frac{\mu_e + \sqrt{\mu_e^2 - m_e^2}}{m_e} \right) \right]. \quad (89)$$

It is neither intuitive nor convenient to express \mathcal{L}_e^T in terms of the chemical potential, it is better to use the difference between the electron and positron number densities

and therefore $\mathcal{L}_e^0 \simeq 0$ when $B = 10B_e$ and, for such values of B , the lightest quark provides the main contribution to the vacuum part of the effective lagrangian \mathcal{L}^0 , which can be written as

$$\sum_f \mathcal{L}_f^0 + \mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0 \simeq \frac{2(eB)^2}{1.215\pi^2} \left(\frac{eB}{m_u^2} \right)^2, \quad (86)$$

where m_u is the u -quark mass. In this scenario, the thermal and density corrections to the effective lagrangian might be much larger than the vacuum part. For example, in the case of a CP -symmetric plasma with $eB \gg T^2$ and $eB \gg m_e^2$, only the lowest Landau levels contributes to \mathcal{L}_e^T and I use Equation (54) to write

$$\mathcal{L}_e^T = \frac{eB}{2\pi^2} \int_0^\infty \frac{dk}{\omega_e} \frac{k^2}{e^{\beta\omega_e} + 1}, \quad (87)$$

where $\omega_e = \sqrt{k^2 + m_e^2}$. An accurate analytical and numerical evaluation of the integral appearing in Equation (87) yields

$$\int_0^\infty \frac{dk}{\omega_e} \frac{k^2}{e^{\beta\omega_e} + 1} \simeq \begin{cases} \frac{\pi^2}{12} T^2 + \frac{m_e^2}{4} \left[\ln \left(\frac{m_e}{\pi T} \right) + \gamma_E - \frac{1}{2} \right] & \text{for } T \geq m_e, \\ \sqrt{\frac{\pi}{2}} T^2 \left(\sqrt{\frac{m_e}{T}} + \frac{1}{6} \right) e^{-\frac{m_e}{T}} & \text{for } T \leq m_e, \end{cases} \quad (88)$$

$n_e - n_{\bar{e}}$, which is related to μ_e by

$$n_e - n_{\bar{e}} = \frac{\partial \mathcal{L}_e^T}{\partial \mu_e}, \quad (90)$$

and therefore in a strong magnetic field it is given by

$$n_e - n_{\bar{e}} = \frac{eB}{2\pi^2} \sqrt{\mu_e^2 - m_e^2}. \quad (91)$$

Using the last relation, I eliminate μ from Equation (89) and rewrite it in the more transparent way

$$\mathcal{L}_e^T = \frac{m_e}{2} (n_e - n_{\bar{e}}) \left[\sqrt{1 + x^2} - \frac{\ln(\sqrt{1 + x^2} + x)}{x} \right], \quad (92)$$

where

$$x = 2\pi^2 \frac{(n_e - n_{\bar{e}})}{eBm_e} \quad (93)$$

is a dimensionless parameter. I find that in a medium with magnetic field $B \sim 10B_e$, temperature $T \ll m_e$ and density $\rho \sim 10^{10} \text{ Kg/m}^3$ the finite density correction to the effective lagrangian obtained from Equation (92), dominates over the vacuum part $\mathcal{L}_e^T \gg \mathcal{L}^0$. These conditions are believed to be common in neutron stars and magnetars. Next I investigate $\mathcal{L}_W^T + \sum_s \mathcal{L}_s^T$, which is the thermal correction to the effective lagrangian from the W and the scalars, and compare it to the vacuum part $\mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0$, in the case of magnetic field $B \ll \frac{M^2}{e}$, where M is the W mass. In this limit, Equation (71) becomes

$$\mathcal{L}_W^T + \sum_s \mathcal{L}_s^T = \frac{1}{\pi^2} \int_0^\infty \frac{dk}{\omega_W} \frac{k^4}{e^{\beta\omega_W} - 1} + \frac{7}{8\pi^2} (eB)^2 \int_0^\infty \frac{dk}{\omega_W} \frac{1}{e^{\beta\omega_W} - 1}, \quad (94)$$

where $\omega_W = \sqrt{k^2 + M^2}$. Notice that the first term, independent of B , is the field independent thermal correction to the lagrangian and that the term with the sum over n of Equation (71) is neglected because, for $eB \ll M^2$, this term is proportional to

$$\frac{(eB)^3}{M^2}.$$

After an accurate evaluation of the integrals appearing in Equation (94) I obtain

$$\int_0^\infty \frac{dk}{\omega_W} \frac{k^4}{e^{\beta\omega_W} - 1} \approx \begin{cases} \frac{\pi^4 T^4}{15} - \frac{\pi^2 T^2 M^2}{4} + \frac{\pi T M^3}{2} & \text{for } T \geq \frac{M}{\sqrt{2}}, \\ 3\sqrt{\frac{\pi}{2}} T^{\frac{5}{2}} M^{\frac{3}{2}} \left[1 + \left(\frac{T}{M}\right)^{\frac{1}{4}} + \frac{2}{3} \left(\frac{T}{M}\right)^{\frac{1}{2}} \right] e^{-\frac{M}{T}} & \text{for } T \leq \frac{M}{\sqrt{2}}, \end{cases} \quad (95)$$

and

$$\int_0^\infty \frac{dk}{\omega_W} \frac{1}{e^{\beta\omega_W} - 1} \approx \begin{cases} \frac{\pi T}{2M} + \frac{1}{2} \left[\ln\left(\frac{M}{4\pi T}\right) + \gamma_E - \frac{1}{2} \right] & \text{for } T \geq \sqrt{10}M, \\ \sqrt{\frac{\pi T}{2M}} \left(1 + \frac{1}{4} \sqrt{\frac{T}{M}} \right) e^{-\frac{M}{T}} & \text{for } T \leq \sqrt{10}M. \end{cases} \quad (96)$$

The maximum error introduced by using (95) and (96) is of a few percent at most. For $eB \ll M^2$ the W and scalar contribution to the vacuum part of the effective lagrangian is obtained easily from Equation (82)

$$\mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0 = \frac{29}{640\pi^2} \frac{(eB)^4}{M^4} \quad (97)$$

and we see that, for

$$T \sim \frac{M}{2 \ln\left(\frac{M^2}{eB}\right)} \quad (98)$$

or higher, the B -dependent term of Equation (94) dominates over $\mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0$, as given by Equation (97). On the other hand, the W and scalars contribution becomes a significant part of the lagrangian only for $T \sim M$ and $m_e^2 \ll eB \ll M^2$.

Last I will investigate the implications of my results on the effective QED coupling in the medium. Due to the scale invariance of eB we can define an effective coupling constant from $\mathcal{L}^{(1)}$ as [10, 15, 17]

$$-\frac{1}{e^2(B, T, \mu)} = \frac{1}{eB} \frac{\partial \mathcal{L}^{(1)}}{\partial (eB)}, \quad (99)$$

from which we obtain the effective electromagnetic fine structure constant

$$\alpha(B, T, \mu) = e^2 \frac{(B, T, \mu)}{4} \pi$$

$$\frac{1}{\alpha(B, T, \mu)} = \frac{1}{\alpha} - \frac{1}{\alpha B} \frac{\partial (\mathcal{L}^0 + \mathcal{L}^T)}{\partial B}. \quad (100)$$

In the limit when $eB = 0$, I find that the effective coupling $\alpha(T, \mu) = \alpha(B = 0, T, \mu)$ is given by

$$\frac{1}{\alpha(T, \mu)} = \frac{1}{\alpha} - \frac{2}{3\pi} \sum_f N_f q_f^2 \int_0^\infty \frac{dk}{\omega_f} [f_F^+(k) + f_F^-(k)] - \frac{7}{2\pi} \int_0^\infty \frac{dk}{\omega_W} \frac{1}{e^{\beta\omega_W} - 1}, \quad (101)$$

with $\omega_f = \sqrt{k^2 + m_f^2}$. When $T = 0$, I find an effective coupling $\alpha(\mu) = \alpha(T = 0, \mu)$ that is

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha} - \frac{2}{3\pi} \sum_f N_f q_f^2 \ln \left(\frac{|\mu_f|}{m_f} + \sqrt{\frac{\mu_f^2}{m_f^2} - 1} \right), \quad (102)$$

dx where μ_f is the chemical potential of the fermion f . Clearly, in an electrically neutral medium at low temperature, the summation will be over e , u , and d only, since only their chemical potential will be relevant.

In a CP-symmetric medium where the temperature is very high but still below the critical temperature of the Weinberg-Salam model, $M \geq T \gg m_b$ where m_b is the mass of the b quark, I find the following behavior of the

corresponding effective coupling $\alpha(T) = \alpha(T, \mu = 0)$

$$\begin{aligned} \frac{1}{\alpha(T)} &= \frac{1}{\alpha} - \frac{2}{3\pi} \sum_{f'} N_{f'} q_{f'}^2 \ln \left(\frac{T}{m_{f'}} \right) \\ &\quad - \frac{16}{9\pi} \int_0^\infty \frac{dk}{\omega_t} \frac{1}{e^{\beta\omega_t} + 1} - \frac{7}{2\pi} \int_0^\infty \frac{dk}{\omega_W} \frac{1}{e^{\beta\omega_W} - 1}, \end{aligned} \quad (103)$$

where the sum is over all fermions except for the quark t and $\omega_t = \sqrt{k^2 + m_t^2}$. The value of the last integral appearing above is given by Equation (96) and, after an accurate analytical and numerical evaluation, I find the value of the other integral

$$\int_0^\infty \frac{dk}{\omega_t} \frac{1}{e^{\beta\omega_t} + 1} = \begin{cases} -\frac{1}{2} \left[\ln \left(\frac{m_t}{\pi T} \right) + \gamma_E \right] & \text{for } T \geq m_t, \\ \sqrt{\frac{\pi T}{2m_t}} \left(1 - \frac{1}{4} \sqrt{\frac{T}{m_t}} \right) e^{-\frac{m_t}{T}} & \text{for } T \leq m_t. \end{cases} \quad (104)$$

Finally I investigate the behavior of $\alpha(B) = \alpha(T = 0, \mu = 0, B)$ when $eB \rightarrow M^2$ and $T = 0, \mu = 0$. Under these conditions, I obtain the following form of the W and scalar contribution to the vacuum effective lagrangian

$$\begin{aligned} \mathcal{L}_W^0 + \sum_s \mathcal{L}_s^0 &\simeq \frac{1}{16\pi^2} \left[\left(\frac{7}{2} - \frac{7}{2} \ln 2 - 12\zeta'(-1) \right) e^2 B^2 \right. \\ &\quad \left. + 2eB(M^2 - eB) \ln \left(\frac{M^2 - eB}{M^2 + eB} \right) \right], \end{aligned} \quad (105)$$

where a constant term proportional to M^4 has been discarded and $\zeta'(-1) = -0.16541$ is the derivative of the Riemann zeta function. Notice the logarithmic branch point in the last term at $eB = M^2$, which indicates that the effective lagrangian picks up an imaginary part when $eB > M^2$ and confirms the well-known result [41–43] concerning the instability of the vacuum. Using Equation (105), I find the following behavior of $\alpha(B)$ when $eB \rightarrow M^2$

$$\frac{1}{\alpha(B)} = \frac{1}{\alpha} - \frac{1}{3\pi} \sum_{f'} N_{f'} q_{f'}^2 \ln \left(\frac{eB}{m_{f'}^2} \right) - \frac{1}{2\pi} \ln \left(\frac{M^2 + eB}{M^2 - eB} \right), \quad (106)$$

where the summation is over all fermions with the exception of the quark t , whose contribution is negligible since it is heavier than the W . The contribution of the W and scalars to the effective coupling constant of Equation (106) is the last term, with the logarithmic branch point at $eB = M^2$.

Appendix

In this appendix I evaluate the Heisenberg-Euler vacuum effective lagrangian and the Weisskopf vacuum effective lagrangian of scalar QED in the case of strong magnetic field $eB \gg m^2$. First I evaluate \mathcal{L}_f^0 , the vacuum effective lagrangian for spinor QED of Equation (40). I introduce a regulator to evaluate the integrals, change the variable of integration from s to $z = eBs$ and write

$$\mathcal{L}_f^0 = -\lim_{\epsilon \rightarrow 0} \frac{(eB)^{2-\epsilon}}{8\pi^2} [I_1(\epsilon, x) - I_2(\epsilon, x)], \quad (A.1)$$

where

$$I_1(\epsilon, x) = \int_0^\infty \frac{dz}{z^{3-\epsilon}} e^{-zx} [z \coth z - 1], \quad (A.2)$$

$$I_2(\epsilon, x) = \frac{1}{3} \int_0^\infty \frac{dz}{z^{1-\epsilon}} e^{-zx} \quad (A.3)$$

and $x = \frac{m^2}{eB}$. When $x \ll 1$, I can take $e^{-zx} \simeq 1$ inside the integral of Equation (A.2), use the following series expansion of the hyperbolic cotangent

$$\coth z = \frac{1}{z} + 2z \sum_{n=1}^\infty \frac{1}{(n\pi)^2 + z^2} \quad (A.4)$$

and find

$$I_1(\epsilon, x) = 2 \sum_{n=1}^\infty \int_0^\infty \frac{dz}{z^{1-\epsilon}} \frac{1}{(n\pi)^2 + z^2}. \quad (A.5)$$

I evaluate this integral and obtain

$$I_1(\epsilon, x) = \pi^{\epsilon-2} \zeta(2-\epsilon) B\left(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}\right), \quad (\text{A.6})$$

where $\zeta(2-\epsilon)$ is the Riemann zeta function and $B\left(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}\right)$ is the Euler beta function. I evaluate $I_2(\epsilon, x)$ exactly and find

$$I_2(\epsilon, x) = \frac{1}{3} x^{-\epsilon} \Gamma(\epsilon), \quad (\text{A.7})$$

where $\Gamma(\epsilon)$ is the Euler gamma function. I insert into Equation (A.1) the values of I_1 and I_2 that I have found, take the limit $\epsilon \rightarrow 0$ and obtain

$$\mathcal{L}_f^0 = -\frac{(eB)^2}{8\pi^2} \left[\frac{1}{3} \ln(\pi x) + \frac{1}{3} \gamma_E - \frac{2}{\pi^2} \zeta'(2) \right], \quad (\text{A.8})$$

where $\gamma_E = 0.5772$ and $\zeta'(2) = -0.9375$ is the first derivative of the Riemann zeta function. Finally, I can significantly simplify Equation (A.8) using the interesting numerical fact [44] that

$$\frac{6}{\pi^2} \zeta'(2) - \log \pi - \gamma_E = -2.2918 \approx -\ln 6 - \frac{1}{2},$$

and write

$$\mathcal{L}_f^0 = \frac{(eB)^2}{24\pi^2} \left[\ln\left(\frac{eB}{6m^2}\right) - \frac{1}{2} \right]. \quad (\text{A.9})$$

I proceed in a similar way when calculating the Weisskopf effective lagrangian of scalar QED for $eB \gg m^2$, where m is the mass of the scalar field. While this result will not be used in this paper, I believe it is of some interest, considering the attention received by the Weisskopf effective lagrangian throughout the years. I start with Equation (67) for \mathcal{L}_s^0 and, after introducing the regulator and changing the integration variable to z , I write

$$\mathcal{L}_s^0 = \lim_{\epsilon \rightarrow 0} \frac{(eB)^{2-\epsilon}}{16\pi^2} \left[I_3(\epsilon, x) + \frac{1}{2} I_2(\epsilon, x) \right], \quad (\text{A.10})$$

where $I_2(\epsilon, x)$ is given in Equation (A.3) and

$$I_3(\epsilon, x) = \int_0^\infty \frac{dz}{z^{3-\epsilon}} e^{-zx} \left[\frac{z}{\sinh z} - 1 \right]. \quad (\text{A.11})$$

After setting $e^{-zx} \simeq 1$ and using the following series expansion of the hyperbolic cosecant

$$\frac{1}{\sinh z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\pi)^2 + z^2}, \quad (\text{A.12})$$

I evaluate $I_3(\epsilon, x)$ and obtain

$$I_3(\epsilon, x) = \pi^{\epsilon-2} (2^{\epsilon-1} - 1) \zeta(2-\epsilon) B\left(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}\right). \quad (\text{A.13})$$

After inserting this value of $I_3(\epsilon, x)$ and the value of $I_2(\epsilon, x)$ obtained previously inside Equation (A.10), I take $\epsilon \rightarrow 0$ and find

$$\mathcal{L}_s^0 = \frac{(eB)^2}{16\pi^2} \left[-\frac{1}{6} \ln\left(\frac{\pi x}{2}\right) - \frac{1}{6} \gamma_E - \frac{1}{\pi^2} \zeta'(2) \right], \quad (\text{A.14})$$

which, using the aforementioned interesting numerical fact, I rewrite as

$$\mathcal{L}_s^0 = \frac{(eB)^2}{96\pi^2} \left[\ln\left(\frac{eB}{3m^2}\right) - \frac{1}{2} \right]. \quad (\text{A.15})$$

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