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Two dimensional fractional projectile motion in a resisting medium

Research Article

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Abstract: In this paper we propose a fractional differential equation describing the behavior of a two dimensional projectile in a resisting medium. In order to maintain the dimensionality of the physical quantities in the system, an auxiliary parameter k was introduced in the derivative operator. This parameter has a dimension of inverse of seconds $(sec)^{-1}$ and characterizes the existence of fractional time components in the given system. It will be shown that the trajectories of the projectile at different values of γ and different fixed values of velocity v_0 and angle θ, in the fractional approach, are always less than the classical one, unlike the results obtained in other studies. All the results obtained in the ordinary case may be obtained from the fractional case when $\gamma = 1$.

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1. Introduction

Fractional calculus (FC) is a branch of mathematics that deals with operators having non-integer(arbitrary) order: fractional derivatives and fractional integrals [1, 4]. In nature, many physical phenomena have "intrinsic" fractional order descriptions [5, 8] and so FC is necessary in order to explain them. FC provides an excellent instrument for the description of memory and hereditary properties of various materials and processes [9]. This is the main advantage of FC in comparison with classical calculus, in which such effects are in fact neglected. Several physical phenomena have been restudied using the tools of fractional calculus, with a great improvement over the traditional analysis as well as better accuracy when comparing with experimental data, for instance, in biomedical, chemical, agricultural and physical applications including Lagrangian and Hamiltonian formulation [10, 15].

In the last few decades FC and fractional differential equations have found applications in various engineering disciplines [16, 20].

Usually, some authors replace integer derivative operators with fractional ones on a purely mathematical or heuristic basis. However, from the physical and engineering point

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of view this is not completely correct because the physical parameters contained in the differential equation should not have the dimensionality measured in the laboratory and the results may not entirely coincide with those obtained in the ordinary case [21, 22].

Recently, in [23, 25] a systematic way to construct fractional differential equations has been proposed, such that the units of the physical parameters involved in the equations remain invariant.

Following [23], in this work we will analyze the behavior of a fractional two dimensional projectile in a resisting medium, when the resistive force is proportional to relative velocity [26]. It will be shown that the trajectories of the projectile at different values of γ and different fixed values of the initial velocity v_0 and angle θ , in the fractional approach, are always less than the classical one, unlike the results obtained in [21].

2. Basic concepts of fractional calculus

To analyze the dynamic behavior of a fractional system it is necessary to use an appropriate definition of the fractional derivative. In many applied problems, it is necessary to use definitions of fractional derivatives that allow the utilization of physically interpretable initial conditions, which contain $x(0), \dot{x}(0), \ldots$. The Caputo representation for fractional order derivatives satisfies these requirements. In the Caputo case, the derivative of a constant is zero, therefore, we can properly define the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer case. As a result, in this manuscript we use the Caputo fractional derivative for a function of time, $f(t)$, defined as [2]

$${}_0^C D_t^\gamma f(t) = \frac{d^\gamma f(t)}{dt^\gamma} = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \quad (1)$$

here $\frac{d^\gamma}{dt^\gamma} = {}_0^C D_t^\gamma$, where $\Gamma(\cdot)$ is the Euler Gamma function, $n = 1, 2, \ldots \in N$ and $n-1 < \gamma \leq n$. Considering the case $n = 1$, i.e., where there is only a first derivative in the integrand, the order of the fractional derivative is $0 < \gamma \leq 1$.

The Laplace transform of Caputo's fractional derivative is given by

$$L\left[{}_0^C D_t^\gamma f(t)\right] = s^\gamma F(s) - \sum_{k=0}^{n-1} s^{\gamma-k-1} f^{(k)}(0). \quad (2)$$

Another definition which will be used is the Mittag-Leffler function. The generalized Mittag-Leffler function is defined as the series expansion

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \qquad (\alpha > 0, \quad \beta > 0), \quad (3)$$

so that $E_{\alpha,1}(z) = E_\alpha(z)$ is the uniparametric Mittag-Leffler function, which is the generalization of the exponential function, $E_1(z) = e^z$ to which it reduces for $\alpha = 1$. Its Laplace transform is given by the formula

$$\int_0^\infty e^{-st} t^{\gamma m+\beta-1} E_{\gamma,\beta}^{(m)}(\pm a t^\gamma) dt = \frac{m! s^{\gamma-\beta}}{(s^\gamma \mp a)^{m+1}}. \quad (4)$$

Consequently, the inverse Laplace transform is

$$L^{-1}\left[\frac{m! s^{\gamma-\beta}}{(s^\gamma \mp a)^{m+1}}\right] = t^{\gamma m+\beta-1} E_{\gamma,\beta}^{(m)}(\pm a t^\gamma). \quad (5)$$

3. Fractional two dimensional motion

Recently, a systematic way to construct fractional differential equations, keeping the unity of the physical parameter invariant, has been proposed in [23]. In this paper we apply this method to analyze the two-dimensional motion of a particle moving in a resisting medium. This problem has been treated in [22], however, there are open questions that could be solved with our method.

The equations of motion of a projectile moving in two-dimensional space with a resistive force proportional to its velocity have the form [26]

$$m\frac{dv_x}{dt} = -kmv_x, \quad (6)$$

$$m\frac{dv_y}{dt} = -mg - kmv_y, \quad (7)$$

where m is the mass of the projectile measured in Kg and k is a real positive constant that specifies the strength of the retarding force; its dimensionality is the inverse of seconds $(sec)^{-1}$. We suppose that the initial conditions are $v_x(0) = v_{0x} = v_0 \cos\theta$ and $v_y(0) = v_{0y} = v_0 \mathrm{sen}\theta$, where v_0 is the initial velocity of the projectile and θ is the angle of elevation. Solutions to Eq.(6) and Eq.(7) are well known [26]

$$x(t) = \frac{v_{0x}}{k}\left(1 - e^{-kt}\right), \quad (8)$$

$$y(t) = -\frac{gt}{k} + \frac{1}{k}\left(v_{0y} + \frac{g}{k}\right)\left[1 - e^{-kt}\right]. \quad (9)$$

Following [23] we replace the ordinary derivative operator $\frac{d}{dt}$ by the fractional operator

$$\frac{d}{dt} \to k^{1-\gamma}\frac{d^\gamma}{dt^\gamma}. \qquad (10)$$

In order to maintain a consistent set of units in equations (6) and (7), the parameter k must have a dimension of the inverse of seconds s^{-1}. Then, we have the following fractional differential equations of order $0 < \gamma \leq 1$

$$\frac{d^\gamma v_x}{dt^\gamma} = -k^\gamma v_x, \qquad (11)$$

$$\frac{d^\gamma v_y}{dt^\gamma} + k^\gamma v_y = -gk^{\gamma-1}. \qquad (12)$$

Using the Laplace transform (2) and the inverse Laplace transform given in (5), equation (11) has the following solution for v_x and $x(t)$

$$v_x(t) = v_{0x}E_\gamma\Big(-(kt)^\gamma\Big), \qquad (13)$$

and

$$x(t) = \frac{v_{0x}}{k}\Big[1 - E_\gamma\Big(-(kt)^\gamma\Big)\Big]. \qquad (14)$$

Using the same procedure for equation (12) we obtain solutions for v_y and $y(t)$

$$v_y(t) = -\frac{g}{k} + \Big[\frac{g}{k} + v_{0y}\Big]E_\gamma\Big(-(kt)^\gamma\Big), \qquad (15)$$

and

$$y(t) = -\frac{g}{k^2\Gamma(\gamma+1)}(kt)^\gamma + \frac{1}{k}\Big(\frac{g}{k} + v_{0y}\Big)\Big[1 - E_\gamma\Big(-(kt)^\gamma\Big)\Big]. \qquad (16)$$

In formulas (13-16) $E_\gamma(\cdot)$ and $\Gamma(\cdot)$ are the Mittag-Leffler function and the Gamma function, respectively. If we set $\gamma = 1$ in formulas (14) and (16) we obtain the formulas for the ordinary case (8) and (9).
Taking into account the series expansion of the Mittag-Leffler function

$$E_\gamma\Big(-(kt)^\gamma\Big) = 1 - \frac{(kt)^\gamma}{\Gamma(\gamma+1)} + \frac{(kt)^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{(kt)^{3\gamma}}{\Gamma(3\gamma+1)} + \ldots, \qquad (17)$$

in formulas (14) and (16), we can plot the trajectory of the projectile for different values of the order of the differential equations $0 < \gamma \leq 1$.
The range R can be found by calculating the time T required for the entire trajectory, and then, substituting this value into equation (14). The time T is found by finding $t = T$, when $y = 0$. From (16) we have

$$(kT)^\gamma = \frac{k\Gamma(\gamma+1)}{g}\Big(\frac{g}{k} + v_{0y}\Big)\Big[1 - E_\gamma\Big(-(kT)^\gamma\Big)\Big]. \qquad (18)$$

This is a transcendental equation and therefore, we cannot obtain an analytic expression for T. However, we can use the expression (17) with $t = T$ and substituting into (18) we obtain

$$(kT)^\gamma = \Gamma(2\gamma+1)\Big[\frac{1}{\Gamma(\gamma+1)} - \frac{g}{\Gamma(\gamma+1)\{g + kv_{0y}\}} + - \frac{(kT)^{2\gamma}}{\Gamma(3\gamma+1)} - \frac{(kT)^{3\gamma}}{\Gamma(4\gamma+1)} + \ldots\Big], \qquad (19)$$

which is the time T required for the entire trajectory. Further substituting this into formula (14) ($t \to T$), we will obtain the range R, given by

$$R = x(T) = \frac{v_{0x}}{k}\Big[1 - E_\gamma\Big\{-(kT)^\gamma\Big\}\Big]. \qquad (20)$$

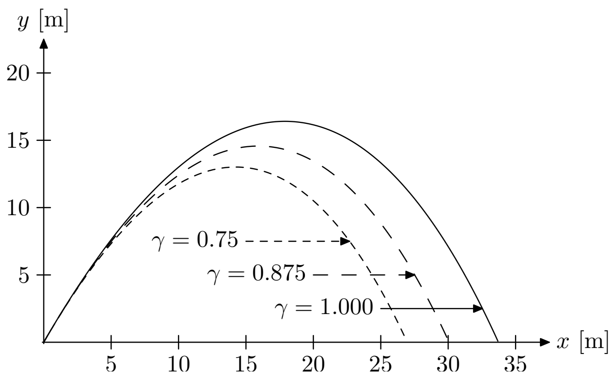

Figure 1. Projectile trajectories obtained from (14) and (16) for different values of γ, $v_0 = 22\,m/s$, $g = 9.81\,m/s^2$, $k = 0.1\,s^{-1}$ and angle $\theta = \pi/3$. The trajectories obtained in the fractional case, never exceed the ordinary ones $\gamma = 1$.

Table 1. Fly time T and range R for different values of γ.

γ	T [sec]	R [m]
1.000	3.661	33.725
0.875	2.6343	30.068
0.750	1.7369	26.9648

4. Conclusion

In this paper we have studied the two dimensional motion of a projectile in a resisting medium from using fractional differential equations. In order to maintain the dimensionality of the physical quantities in the system, an auxiliary parameter k was introduced in the derivative operator. This parameter has a dimension of inverse of seconds $(sec)^{-1}$ and characterizes the existence of fractional time components in the given system.

It was shown that whenever inequalities $v_0 \sin\theta > 0$ and $\theta < \pi/2$ are fulfilled, the trajectories of the projectile at different values of γ and different fixed values of velocity v_0 calculated using the fractional approach, are always less than the classical approach. This is unlike the results obtained in other studies [22]. All the results obtained in the ordinary case may be obtained from the fractional case when $\gamma = 1$. The extension to three dimensional motion may be done with this method.

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