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Erlang Strength Model for Exponential Effects

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Abstract: All technical systems have been designed to perform their intended tasks in a specific ambient. Some systems can perform their tasks in a variety of distinctive levels. A system that can have a finite number of performance rates is called a multi-state system. Generally multi-state system is consisted of components that they also can be multi-state. The performance rates of components constituting a system can also vary as a result of their deterioration or in consequence of variable environmental conditions. Components failures can lead to the degradation of the entire multi-state system performance. The performance rates of the components can range from perfect functioning up to complete failure. The quality of the system is completely determined by components. In this article, a possible state for the single component system, where component is subject to two stresses, is considered under stress-strength model which makes the component multi-state. The probabilities of component are studied when strength of the component is Erlang random variables and the stresses are independent exponential random variables. Also, the probabilities of component are considered when the stresses are dependent exponential random variables.

Keywords: Reliability; Stress-Strength Model; Multi-State System; Erlang Distribution; Exponential Distribution

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1 Introduction

In a multistate system model, a system and its components may be in $M + 1$ possible states $0, 1, 2, \dots, M$, where 0 indicates the completely failed state, M indicates the perfectly working state, and others degraded states. In other

words, the state space of each component and the system is taken to be $\{0, 1, \dots, M\}$, where M is a positive integer. Actually, we have a discrete multi-state system model because the state of each component and the system can be represented by integer numbers. Indeed, a binary system is the simplest case of a multi-state system having two distinguished states; perfect functioning and completely failure.

In a binary system, the definition domains of the states of the system and its components are $\{0, 1\}$. Multi-state systems have been found to be more flexible tool than binary systems for modeling engineering systems. In literature, much attention has been paid to multi-state system modeling ([3],[5],[6],[8],[12],[14],[19]).

For reliability analysis, stress-strength models are of special importance. In the simplest terms, stress-strength model can be described as an assessment of the reliability of the component in terms of X and Y random variables where Y is the random “stress” experienced by the component and X is the random “strength” of the component available to overcome the stress. From this simplified explanation, the reliability of the component is the probability that the component is strong enough to overcome the stress applied on it, i.e., $R = P(Y < X)$.

Extensive works have been done for the reliability of the component and its estimation under different choices for stress and strength distributions ([1],[2],[4],[7],[9],[10],[13],[17],[18]).

In this article, we consider following three possible states for a single component system under stress-strength model which makes the component multistate. Suppose that there is a system consisting of Z prone to failure independent component, Y_1 and Y_2 are random stresses experienced by the component and X is the random strength of the component to overcome the stresses.

Complete failure state.

$Z = 0$ if the strength of the component does not exceed both of the stresses Y_1 and Y_2 .

That is, $X < Y_{1:2} = \min(Y_1, Y_2)$.

Lower state.

$Z = 1$ if the strength of the component exceeds only one stresses Y_1 and Y_2 .

That is, $Y_{1:2} < X < Y_{2:2} = \max(Y_1, Y_2)$.

Perfect state.

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$Z = 2$ if the strength of the component exceeds both of the stresses Y_1 and Y_2 . That is, $X > Y_{2:2}$.

In a multistate system, state distribution is often used to indicate the distribution of each component in different state. The performance of the component is represented in terms of its state distribution. Therefore, the state distribution is the most important performance measure in a multistate system. On the other hand, a state distribution can be given by a probability distribution function, a cumulative distribution function or a reliability function.

Let us assume that strength of the component X is independent identically distributed with continuous cumulative distribution function $F(z) = P\{X \leq z\}$ and the stresses Y_1 and Y_2 are independent random variables having continuous cumulative distribution function $G_l(z) = P\{Y_l \leq z\}$, $l = 1, 2$.

In the following, we obtain the probability of being in three possible states, as mentioned above, for a component in a system (see [9]).

The probability of being in complete failure state for a component is obtained as follows:

$$p_0 = P\{Z = 0\} = \int_{R_y} P\{X < Y_{1:2}, Y_{1:2} = y\} dy$$

Conditioning on $Y_{1:2} = y$, we have

$$p_0 = \int_{R_y} F(y) dP\{Y_{1:2} \leq y\}, \tag{1}$$

where $P\{Y_{1:2} \leq y\} = 1 - \bar{G}_1(y)\bar{G}_2(y)$ and $\bar{G} = 1 - G$.

The probability of being in lower state for a component is obtained as follows:

$$p_1 = P\{Z = 1\} = \iint_{y_1 < y_2} P\{Y_{1:2} < X < Y_{2:2}, Y_{1:2} = y_1; Y_{2:2} = y_2\} dy_1 dy_2$$

Conditioning on $Y_{1:2} = y_1, Y_{2:2} = y_2$ we have

$$p_1 = \iint_{y_1 < y_2} (F(y_2) - F(y_1)) dP\{Y_{1:2} \leq y_1, Y_{2:2} \leq y_2\}, \tag{2}$$

where

$$P\{Y_{1:2} \leq y_1, Y_{2:2} \leq y_2\} = G_1(y_2)G_2(y_2) - (G_1(y_2) - G_1(y_1))(G_2(y_2) - G_2(y_1))$$

Similarly, the probability of being in perfect state for a component is obtained as follows:

$$p_2 = P\{Z = 2\} = \int_{R_y} \bar{F}(y) dP\{Y_{2:2} \leq y\}, \tag{3}$$

where $P\{Y_{2:2} \leq y\} = G_1(y)G_2(y)$ and $\bar{F} = 1 - F$.

In the present paper, in section 2, the probability that a component is in three possible states are studied under the strength-stress model when strength of the component is Erlang random variable and the stresses are independent exponential random variables with different parameters. In section 3, we consider the probability that a component is in three possible states when the stresses are dependent exponential random variables. In the last section, we summarize what we have done in the article and give some conclusions.

We will use the following special function and integral in the next section to establish the component states. The function is the complementary incomplete gamma function defined by

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt. \tag{4}$$

The integral is

$$\int_0^\infty e^{-ax} \Gamma(\beta, x) dx = \frac{1}{\alpha} \Gamma(\beta) \left[1 - \frac{1}{(\alpha + 1)^\beta} \right] \tag{5}$$

where $\beta > 0$ and Γ is the gamma function (Eq. 6.451.2 in [11]).

2 State distributions for independent stresses

In this section, we will consider the strength of the component is Erlang random variable and the stresses are independent exponential random variables with different parameters. That is, for $z \geq 0$, the cumulative distribution function of X and Y_1, Y_2 are, respectively,

$$F(z) = 1 - \sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{z}{\theta}\right)^m e^{-\frac{z}{\theta}}$$

and

$$G_l(z) = 1 - e^{-\frac{z}{\theta_l}},$$

where k positive integer and $\theta, \theta_l > 0$.

Now using (1)-(3), component's three possible states can be computed as follows:

Complete failure state.

$$p_0 = \int_0^\infty \left(1 - \sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{y}{\theta} \right)^m e^{-\frac{y}{\theta}} \right) \left(\frac{1}{\theta_2} e^{-\frac{y}{\theta_2}} e^{-\frac{y}{\theta_1}} + \frac{1}{\theta_1} e^{-\frac{y}{\theta_1}} e^{-\frac{y}{\theta_2}} \right) dy = 1 - \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right) \sum_{m=1}^k \frac{1}{\theta^{m-1} \left(\frac{1}{\theta} + \frac{1}{\theta_1} + \frac{1}{\theta_2} \right)^m}$$

Lower state.

$$\begin{aligned} p_1 &= \frac{1}{\theta_1 \theta_2} \sum_{m=0}^{k-1} \frac{1}{m! \theta^m} \int_0^\infty \int_{y_1}^\infty \left(y_1^m e^{-\frac{y_1}{\theta}} - y_2^m e^{-\frac{y_2}{\theta}} \right) \left(e^{-\frac{y_2}{\theta_1}} e^{-\frac{y_1}{\theta_2}} + e^{-\frac{y_1}{\theta_1}} e^{-\frac{y_2}{\theta_2}} \right) dy_2 dy_1 \\ &= \frac{1}{\theta_1 \theta_2} \sum_{m=1}^k \frac{1}{(m-1)! \theta^{m-1}} \int_0^\infty \left[y_1^{m-1} e^{-y_1 \left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)} \int_{y_1}^\infty e^{-\frac{y_2}{\theta_1}} dy_2 + y_1^{m-1} e^{-y_1 \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)} \right. \\ &\quad \left. \times \int_{y_1}^\infty e^{-\frac{y_2}{\theta_2}} dy_2 - e^{-\frac{y_1}{\theta_2}} \int_{y_1}^\infty y_2^{m-1} e^{-y_2 \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)} dy_2 - e^{-\frac{y_1}{\theta_1}} \int_{y_1}^\infty y_2^{m-1} e^{-y_2 \left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)} dy_2 \right] dy_1 \\ &= \frac{1}{\theta_1 \theta_2} \sum_{m=1}^k \frac{1}{(m-1)! \theta^{m-1}} \left[(\theta_1 + \theta_2) \int_0^\infty y_1^{m-1} e^{-y_1 \left(\frac{1}{\theta} + \frac{1}{\theta_1} + \frac{1}{\theta_2} \right)} dy_1 - \frac{1}{\left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)^m} \int_0^\infty e^{-\frac{y_1}{\theta_2}} \Gamma \left(m, y_1 \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right) \right) dy_1 \right. \\ &\quad \left. - \frac{1}{\left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)^m} \int_0^\infty e^{-\frac{y_1}{\theta_1}} \Gamma \left(m, y_1 \left(\frac{1}{\theta} + \frac{1}{\theta_2} \right) \right) dy_1 \right] \\ &= \sum_{m=1}^k \frac{1}{\theta^{m-1}} \left[\frac{\frac{1}{\theta_1} + \frac{1}{\theta_2}}{\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \frac{1}{\theta_2} \right)^m} - \frac{\frac{1}{\theta_1}}{\left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)^m} \left(1 - \left(\frac{\theta_2 \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)}{1 + \theta_2 \left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)} \right)^m \right) \right. \\ &\quad \left. - \frac{\frac{1}{\theta_2}}{\left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)^m} \left(1 - \left(\frac{\theta_1 \left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)}{1 + \theta_1 \left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)} \right)^m \right) \right], \end{aligned}$$

where suitable transformations and simplifications have been applied and (4)-(5) used.

Perfect state.

$$\begin{aligned} p_2 &= \int_0^\infty \left(\sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{y}{\theta} \right)^m e^{-\frac{y}{\theta}} \right) \left(\frac{1}{\theta_1} e^{-\frac{y}{\theta_1}} \left(1 - e^{-\frac{y}{\theta_2}} \right) + \frac{1}{\theta_2} e^{-\frac{y}{\theta_2}} \left(1 - e^{-\frac{y}{\theta_1}} \right) \right) dy \\ &= \sum_{m=1}^k \frac{1}{\theta^{m-1}} \left(\frac{\frac{1}{\theta_1}}{\left(\frac{1}{\theta} + \frac{1}{\theta_1} \right)^m} + \frac{\frac{1}{\theta_2}}{\left(\frac{1}{\theta} + \frac{1}{\theta_2} \right)^m} - \frac{\frac{1}{\theta_1} + \frac{1}{\theta_2}}{\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \frac{1}{\theta_2} \right)^m} \right) \end{aligned}$$

Table 1 and 2 contain some numerical results for component's three possible states for selected values of the parameters k, θ, θ_1 and θ_2 .

As shown in this section, the stresses are independent random variables. However, from practical viewpoint, the stresses may be dependent random variables. Now let the stresses are dependent, that is, the random variables Y_1 and Y_2 have the joint cumulative distribution function $G(z_1, z_2)$ and probability density function $g(z_1, z_2)$. Then, the distributions of $Y_{1:2}$ and $Y_{2:2}$ are obtained to be:

$$P \{ Y_{1:2} \leq y \} = 1 - \bar{G}(y, y) = G_1(y) + G_2(y) - G(y, y), \tag{6}$$

Table 1: Probabilities of component states with same parameters.

k	θ	θ_1	θ_2	p_0	p_1	p_2
1	0.1	0.1	0.1	0.333	0.333	0.333
2	0.1	0.1	0.1	0.111	0.278	0.611
3	0.1	0.1	0.1	0.037	0.176	0.787
4	0.1	0.1	0.1	0.012	0.101	0.887
5	0.1	0.1	0.1	0.004	0.055	0.941

Table 2: Probabilities of component states with different parameters.

k	θ	θ_1	θ_2	p_0	p_1	p_2
1	0.1	0.2	0.3	0.1666	0.7047	0.1287
2	0.1	0.2	0.3	0.0277	0.6818	0.2905
3	0.1	0.2	0.3	0.0046	0.5513	0.4441
4	0.1	0.2	0.3	0.0007	0.4248	0.5745
5	0.1	0.2	0.3	0.0001	0.3207	0.6792

$$P\{Y_{1:2} \leq y_1, Y_{2:2} \leq y_2\} = G(y_1, y_2) + G(y_2, y_1) - G(y_1, y_1), \tag{7}$$

$$P\{Y_{2:2} \leq y\} = G(y, y), \tag{8}$$

where $\bar{G}(y, y) = P\{Y_1 > y_1, Y_2 > y_2\}$ denotes the bivariate survival function. The state probabilities will be computed in the following section using (6)-(8).

3 State distributions for dependent stresses

Let (Y_1, Y_2) have a bivariate exponential distribution with survival function by

$$\bar{G}(y_1, y_2) = e^{-\lambda y_1 - \beta y_2 - \theta_1 \max\{y_1, y_2\}} \quad y_1, y_2 > 0,$$

where $\lambda, \beta, \theta_1 > 0$ ([15]) and the strength of the component have gamma distribution.

Now using (6)-(8) in (1)-(3), component's three possible states can be computed as follows:

Complete failure state.

$$p_0 = \int_0^\infty \left(1 - \sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{y}{\theta}\right)^m e^{-\frac{y}{\theta}} \right) \left((\lambda + \beta + \theta_1) e^{-y(\lambda + \beta + \theta_1)} \right) dy$$

$$= 1 - (\lambda + \beta + \theta_1) \sum_{m=1}^k \frac{1}{\theta^{m-1} \left(\frac{1}{\theta} + \lambda + \beta + \theta_1\right)^m}$$

Table 3: Probabilities of component states with same parameters.

k	θ	λ	β	θ_1	p_0	p_1	p_2
1	0.1	0.1	0.1	0.1	0.970	0.020	0.010
2	0.1	0.1	0.1	0.1	0.942	0.047	0.035
3	0.1	0.1	0.1	0.1	0.915	0.074	0.073
4	0.1	0.1	0.1	0.1	0.888	0.101	0.109
5	0.1	0.1	0.1	0.1	0.862	0.127	0.146

Lower state.

$$p_1 = \sum_{m=0}^{k-1} \frac{1}{m! \theta^m} \int_0^\infty \int_0^\infty \left(y_1^m e^{-\frac{y_1}{\theta}} - y_2^m e^{-\frac{y_2}{\theta}} \right) \times \left(\lambda(\beta + \theta_1) e^{-y_1 \lambda - y_2(\beta + \theta_1)} + \beta(\lambda + \theta_1) e^{-y_1 \beta - y_2(\lambda + \theta_1)} \right) dy_2 dy_1$$

$$= \sum_{m=1}^k \frac{1}{\theta^{m-1}} \left[\frac{\lambda + \beta}{\left(\frac{1}{\theta} + \lambda + \beta + \theta_1\right)^m} - \frac{\beta + \theta_1}{\left(\frac{1}{\theta} + \beta + \theta_1\right)^m} \left(1 - \left(\frac{\frac{1}{\theta} + \beta + \theta_1}{\frac{1}{\theta} + \lambda + \beta + \theta_1}\right)^m \right) - \frac{\lambda + \theta_1}{\left(\frac{1}{\theta} + \lambda + \theta_1\right)^m} \left(1 - \left(\frac{\frac{1}{\theta} + \lambda + \theta_1}{\frac{1}{\theta} + \lambda + \beta + \theta_1}\right)^m \right) \right]$$

Perfect state.

$$p_2 = \int_0^\infty \left(\sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{y}{\theta}\right)^m e^{-\frac{y}{\theta}} \right) \left((\lambda + \theta_1) e^{-y(\lambda + \theta_1)} + (\beta + \theta_1) e^{-y(\beta + \theta_1)} - (\lambda + \beta + \theta_1) e^{-y(\lambda + \beta + \theta_1)} \right) dy$$

$$= \sum_{m=1}^k \frac{1}{\theta^{m-1}} \left(\frac{\lambda + \theta_1}{\left(\frac{1}{\theta} + \lambda + \theta_1\right)^m} + \frac{\beta + \theta_1}{\left(\frac{1}{\theta} + \beta + \theta_1\right)^m} - \frac{\lambda + \beta + \theta_1}{\left(\frac{1}{\theta} + \lambda + \beta + \theta_1\right)^m} \right)$$

Table 3 and 4 contain some numerical results for component's three possible states for selected values of the parameters $k, \theta, \lambda, \beta$ and θ_1 .

4 Conclusions

When the issue of reliability of technical systems concerns the many branch of the technology, in particular first industrial engineering, the techniques which help to investigate the issue entirely comes from the basic concepts of possibility and statistical analysis. From this perspective,

Table 4: Probabilities of component states with different parameters.

k	θ	λ	β	θ_1	p_0	p_1	p_2
1	0.1	0.2	0.3	0.4	0.917	0.043	0.039
2	0.1	0.2	0.3	0.4	0.841	0.080	0.078
3	0.1	0.2	0.3	0.4	0.772	0.111	0.116
4	0.1	0.2	0.3	0.4	0.708	0.138	0.153
5	0.1	0.2	0.3	0.4	0.649	0.160	0.189

technical systems can be seen as a natural application area for probability and statistical analysis.

When taking into consideration the development of the literature, it is configured firstly as the system connections divided two main groups connected series or parallel and later as the k -out-of- n system in terms of ease of application. Structurally, the optimal operation of the system depends on the optimal operation of the components which constitute the system. However we have some questions, which part is the malfunction or which part isn't the full performance and how much these will affect the work of the system? The answer to this problem is connected directly to how components combined with different connection models. In this study, we have focused on a single component which constitutes the system and examined the strength of the component under the stress model. In the similar studies existing in the literature, the strength of the components is connected to the continuous probability distributions. This situation leads to the breaking at a point strength of machine working under the pressure and does not allow optimization of the working. In this investigation, it's evaluated as the two-phase factor affecting the working of machines and accordingly, expressing the working of machines is selected from the Erlang distribution which is phase-type distribution. When the factors of stresses were taken two exponential distributions, independent and with different parameters, the optimal operation of the machine is possible to increase the phase number of Erlang distribution. The similar situation is also observed in the dependent impacts. As the distribution parameters of the effect made the machine increase, providing the optimal operation may be possible with two ways. The first of these is to increase the strength parameter and the second is to increase the phase number of the distribution. The strength parameters of machine running and observed is constant. However, in case of the increase of the phase number and the exponential of strength model, supporting the spare machines may be possible the optimal operation. In Erlang strength model, the optimal operation can be achieved when by increasing the parameter

k or/and by connecting to be the backup each other, which k number of machine, working with the same exponential strength model. This result is an important optimality criterion for the reliability of a technical system. In this respect, the study brings a different perspective to the subject.

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